Mathematics 210
Homework 1
Answers

1. Use row reduction to find the solution set of this system of equations:

$$\begin{aligned} 6x + 7y + 4z &= 89 \\ x + 2y + 3z &= 40\,. \\ 2x + 3y + z &= 30 \end{aligned}$$

Answer: Writing an arrow to indicate any elementary row operation, and occasionally simplifying some arithmetic by multiplying a row by a constant, we have

$$\begin{bmatrix} 6 & 7 & 4 & 89 \\ 1 & 2 & 3 & 40 \\ 2 & 3 & 1 & 30 \end{bmatrix} \to \begin{bmatrix} 6 & 7 & 4 & 89 \\ 0 & \frac{5}{6} & \frac{7}{3} & \frac{151}{6} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 6 & 7 & 4 & 89 \\ 0 & 1 & \frac{14}{5} & \frac{151}{5} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 6 & 7 & 4 & 89 \\ 0 & 1 & \frac{14}{5} & \frac{151}{5} \\ 0 & 0 & -\frac{11}{5} & -\frac{99}{5} \end{bmatrix} \to$$
$$\begin{bmatrix} 6 & 7 & 4 & 89 \\ 0 & 1 & \frac{14}{5} & \frac{151}{5} \\ 0 & 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 6 & 7 & 0 & 53 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 6 & 0 & 0 & 18 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

We can now read off the solution: $x_1 = 3$, $x_2 = 5$, and $x_3 = 9$.

2. Use row reduction to find the solution set of this system of equations:

$$\begin{aligned} 9x_1 + 4x_2 + 5x_3 &= 115 \\ 3x_1 + 2x_2 - 4x_3 &= 98\,. \\ 5x_1 + 4x_2 + 5x_3 &= 67 \end{aligned}$$

Answer: Again, we use an arrow for any elementary row operation:

$$\begin{bmatrix} 9 & 4 & 5 & 115 \\ 3 & 2 & -4 & 98 \\ 5 & 4 & 5 & 67 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 5 & 115 \\ 0 & \frac{2}{3} & -\frac{17}{3} & \frac{179}{3} \\ 0 & \frac{16}{9} & \frac{20}{9} & \frac{28}{9} \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 5 & 115 \\ 0 & \frac{2}{3} & -\frac{17}{3} & \frac{179}{3} \\ 0 & 0 & \frac{52}{3} & -156 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 5 & 115 \\ 0 & \frac{2}{3} & -\frac{17}{3} & \frac{179}{3} \\ 0 & 0 & 1 & -9 \end{bmatrix} \to$$
$$\begin{bmatrix} 9 & 4 & 0 & 160 \\ 0 & \frac{2}{3} & 0 & \frac{26}{3} \\ 0 & 0 & 1 & -9 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 0 & 160 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -9 \end{bmatrix} \to \begin{bmatrix} 9 & 0 & 0 & 108 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

Therefore, the solution is $x_1 = 12$, $x_2 = 13$, and $x_3 = -9$.

3. Can you find values of a, b, and c that make this augmented matrix

$$\begin{bmatrix} 4 & 5 & a & 3 \\ 6 & 5 & b & 3 \\ 2 & 2 & c & 11 \end{bmatrix}$$

correspond to an *inconsistent* system of equations?

Answer: There are many particular values of a, b, and c which will make the matrix correspond to an inconsistent system of equations. To find the general solution, we row-reduce the matrix:

$$\begin{bmatrix} 4 & 5 & a & 3 \\ 6 & 5 & b & 3 \\ 2 & 2 & c & 11 \end{bmatrix} \to \begin{bmatrix} 4 & 5 & a & 3 \\ 0 & -\frac{5}{2} & b - \frac{3}{2}a & -\frac{3}{2} \\ 0 & -\frac{1}{2} & c - \frac{1}{2}a & \frac{19}{2} \end{bmatrix} \to \begin{bmatrix} 4 & 5 & a & 3 \\ 0 & -\frac{5}{2} & b - \frac{3}{2}a & -\frac{3}{2} \\ 0 & 0 & c - \frac{a}{5} - \frac{b}{5} & \frac{49}{5} \end{bmatrix}$$

This matrix will lead to an inconsistent set of equations only when $c - \frac{a}{5} - \frac{b}{5} = 0$, in which case the final row will lead to the equation $0 = \frac{40}{5}$. So the initial matrix will lead to an inconsistent set of equations whenever $5c = a + b$.

4. Consider this system of equations:

$$\begin{aligned} x_1 + 3x_2 &= c_1 \\ ax_1 + bx_2 &= c_2 \end{aligned},$$

where a, b, c_1, and c_2 are unknown fixed real numbers. If you are told that this is a consistent set of equations for all values of c_1 and c_2, what (if anything) can you say about the values of the unknowns a and b?

Answer: We row-reduce the augmented matrix:

$$\begin{bmatrix} 1 & 3 & c_1 \\ a & b & c_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & c_1 \\ 0 & b-3a & c_2 - ac_1 \end{bmatrix}$$

The system of equations is consistent as long as $b - 3a \neq 0$, so we can say that $b \neq 3a$.

5. Consider these two augmented matrices:

$$\begin{bmatrix} 3 & 4 & 11 \\ 4 & 3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 5 & 6 & 13 \\ 6 & 5 & 13 \end{bmatrix}$$

Do they correspond to equivalent systems of equations?

Answer: By doing a bit of algebra, we see that the first system has the solution set $x_1 = \frac{15}{7}$ and $x_2 = \frac{8}{7}$, while the second system has the solution set $x_1 = x_2 = \frac{13}{11}$. Because the two systems of equations have different solution sets, they are not equivalent.

6. Find the solution set of the system corresponding to this augmented matrix:

$$\begin{bmatrix} 3 & 4 & 5 & 11 \\ 2 & 3 & 4 & 12 \end{bmatrix}$$

Answer: Row reduction yields:

$$\begin{bmatrix} 3 & 4 & 5 & 11 \\ 2 & 3 & 4 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 & 5 & 11 \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{14}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 & 5 & 11 \\ 0 & 1 & 2 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & -3 & -45 \\ 0 & 1 & 2 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -15 \\ 0 & 1 & 2 & 14 \end{bmatrix}$$

This corresponds to the two equations $x_1 - x_3 = -15$ and $x_2 + 2x_3 = 14$. Therefore, x_3 is a free variable, x_1 and x_2 are basic variables, and the parametric form of the solution is

$$\begin{aligned} x_1 &= -15 + x_3 \\ x_2 &= 14 - 2x_3 \end{aligned}.$$

7. Find the solution set of the system corresponding to this augmented matrix:

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ -1 & 7 & 4 & 2 & 7 \end{bmatrix}$$

Answer: We have

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ -1 & 7 & 4 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 8 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the two equations $x_1 - 7x_2 + 6x_4 = 5$ and $x_3 + 2x_4 = 3$. The free variables are x_2 and x_4, and the basic variables are x_1 and x_3. A parametric form of the solution is

$$\begin{aligned} x_1 &= 5 + 7x_2 - 6x_4 \\ x_3 &= 3 - 2x_4 \end{aligned}$$

8. Find a system of equations that is equivalent to this vector equation:

$$x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \\ -3 \end{bmatrix}.$$

You do not need to find the solution set of the system.

Answer: An equivalent system of equations is:

$$\begin{array}{rcrcr} 6x_1 & + & 3x_2 & = & 11 \\ -x_1 & + & 4x_2 & = & 12 \\ 5x_1 & & & = & -3 \end{array}.$$

9. Find a vector equation which is equivalent to this system:

$$\begin{aligned} 3x_2 - 4x_3 &= 11 \\ 2x_1 + 4x_2 + 9x_3 &= 13 \end{aligned}.$$

Answer: One solution is $x_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$.

10. Let

$$A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ -2 & 6 & 3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}.$$

Is $\mathbf{b}$ a linear combination of the vectors formed by the columns of A?

Answer: We are asking if the equation $x_1 \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$ has any solutions. We row-reduce the augmented matrix:

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ -2 & 6 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & 6 & 9 & 13 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 6 & 10 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 3 & 7 \end{bmatrix}$$

Because this is matrix corresponds to a consistent set of equations, we know that the original system has a solution, and therefore $\mathbf{b}$ is indeed a linear combination of the columns of A.