Mathematics 210
Homework 2
Answers

1. Find 4 vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b} \in \mathbf{R}^3$, all non-zero and all unequal, so that $\mathbf{b}$ is *not* in the span of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Answer: Obviously, there are many possible ways to answer this question. One answer is $\mathbf{a}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}2\\2\\2\end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix}3\\3\\3\end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

2. Compute

$$\begin{bmatrix}1 & 2\\4 & 5\\-1 & 11\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}.$$

Answer: We have

$$\begin{bmatrix}1 & 2\\4 & 5\\-1 & 11\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}11\\32\\41\end{bmatrix}$$

3. Compute

$$\begin{bmatrix}1 & 2 & 4\\5 & -1 & 11\end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix}.$$

Answer: We have

$$\begin{bmatrix}1 & 2 & 4\\5 & -1 & 11\end{bmatrix}\begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}31\\66\end{bmatrix}.$$

4. Is $\begin{bmatrix}1\\1\\3\end{bmatrix}$ in the span of $\begin{bmatrix}4\\5\\7\end{bmatrix}$, $\begin{bmatrix}3\\11\\9\end{bmatrix}$ and $\begin{bmatrix}3\\10\\9\end{bmatrix}$?

Answer: This is the same question as asking whether the system $\begin{bmatrix}4 & 3 & 3\\5 & 11 & 10\\7 & 9 & 9\end{bmatrix}\mathbf{x} = \begin{bmatrix}1\\1\\3\end{bmatrix}$ has a solution. We proceed with row reduction:

$$\begin{bmatrix}4 & 3 & 3 & 1\\5 & 11 & 10 & 1\\7 & 9 & 9 & 3\end{bmatrix} \to \begin{bmatrix}4 & 3 & 3 & 1\\0 & \frac{29}{4} & \frac{25}{4} & -\frac{1}{4}\\0 & \frac{15}{4} & \frac{15}{4} & \frac{5}{4}\end{bmatrix} \to \begin{bmatrix}4 & 3 & 3 & 1\\0 & \frac{29}{4} & \frac{25}{4} & -\frac{1}{4}\\0 & 0 & \frac{15}{29} & \frac{40}{29}\end{bmatrix}$$

Because each column is a pivot, we know that there is a unique solution, so indeed $\begin{bmatrix}1\\1\\3\end{bmatrix}$ is in the span of $\begin{bmatrix}4\\5\\7\end{bmatrix}$, $\begin{bmatrix}3\\11\\9\end{bmatrix}$ and $\begin{bmatrix}3\\10\\9\end{bmatrix}$.

5. Can every vector in $\mathbf{R}^2$ be written as a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}3\\4\end{bmatrix}$, and $\begin{bmatrix}5\\6\end{bmatrix}$?

Answer: We apply row reduction: $\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix} \to \begin{bmatrix}1&3&5\\0&-2&-4\end{bmatrix}$. Because each row has a pivot, that suffices to tell us there is never a possibility of an inconsistent system of equations, and therefore $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}3\\4\end{bmatrix}$, and $\begin{bmatrix}5\\6\end{bmatrix}$ span $\mathbf{R}^2$.

6. Can every vector in $\mathbf{R}^3$ be written as a linear combination of $\begin{bmatrix}7\\8\\9\end{bmatrix}$, $\begin{bmatrix}10\\11\\12\end{bmatrix}$, and $\begin{bmatrix}13\\14\\15\end{bmatrix}$?

Answer: Again we apply row reduction:

$$\begin{bmatrix}7&10&13\\8&11&14\\9&12&15\end{bmatrix} \to \begin{bmatrix}7&10&13\\0&-\frac{3}{7}&-\frac{6}{7}\\0&-\frac{6}{7}&-\frac{12}{7}\end{bmatrix} \to \begin{bmatrix}7&10&13\\0&-\frac{3}{7}&-\frac{6}{7}\\0&0&0\end{bmatrix}$$

This shows that not every vector in $\mathbf{R}^3$ is in the span of $\begin{bmatrix}7\\8\\9\end{bmatrix}$, $\begin{bmatrix}10\\11\\12\end{bmatrix}$, and $\begin{bmatrix}13\\14\\15\end{bmatrix}$.

7. Let $A = \begin{bmatrix}2&3&4&7\\12&11&9&6\end{bmatrix}$. Describe the solution set of the system of equations $A\mathbf{x} = \mathbf{0}$ in parametric form.

Answer: We can omit the column of 0's from the augmented matrix, remembering to use it at the end of the computation. We row reduce A: $\begin{bmatrix}2&3&4&7\\12&11&9&6\end{bmatrix} \to \begin{bmatrix}2&3&4&7\\0&-7&-15&-36\end{bmatrix} \to$ $\begin{bmatrix}2&3&4&7\\0&1&\frac{15}{7}&\frac{36}{7}\end{bmatrix} \to \begin{bmatrix}2&0&-\frac{17}{7}&-\frac{59}{7}\\0&1&\frac{15}{7}&\frac{36}{7}\end{bmatrix} \to \begin{bmatrix}1&0&-\frac{17}{14}&-\frac{59}{14}\\0&1&\frac{15}{7}&\frac{36}{7}\end{bmatrix}$. In equations, we have $x_1 =$

$\frac{17}{14}x_3 + \frac{59}{14}x_4$ and $x_2 = -\frac{15}{7}x_3 - \frac{36}{7}x_4$. So our solution looks like

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{17}{14}x_3 + \frac{59}{14}x_4 \\ -\frac{15}{7}x_3 - \frac{36}{7}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{17}{14} \\ -\frac{15}{7} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{59}{14} \\ -\frac{36}{7} \\ 0 \\ 1 \end{bmatrix}$$

The next two problems are designed to prove Theorem 6 in Chapter 1.

8. Suppose that $\mathbf{p}$ solves the equation $A\mathbf{x} = \mathbf{b}$. Suppose that $\mathbf{v}_h$ solves the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$. Show that $\mathbf{w}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}$.
Answer: We have $A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

9. Suppose that $\mathbf{w}$ and $\mathbf{p}$ are two solutions of the equation $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{v}_h = \mathbf{w} - \mathbf{p}$. Show that $\mathbf{v_h}$ solves the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
Answer: We have $A\mathbf{v}_h = A(\mathbf{w} - \mathbf{p}) = A\mathbf{w} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

10. Let

$$A = \begin{bmatrix} 1 & 2 & 11 \\ 3 & 5 & 10 \\ 7 & 6 & 9 \\ 4 & 3 & 2 \end{bmatrix}.$$

Are the columns of A linearly independent vectors?
Answer: As usual, we row-reduce:

$$\begin{bmatrix} 1 & 2 & 11 \\ 3 & 5 & 10 \\ 7 & 6 & 9 \\ 4 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 11 \\ 0 & -1 & -23 \\ 0 & -8 & -68 \\ 0 & -5 & -42 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 11 \\ 0 & -1 & -23 \\ 0 & 0 & 116 \\ 0 & 0 & 73 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 11 \\ 0 & -1 & -23 \\ 0 & 0 & 116 \\ 0 & 0 & 0 \end{bmatrix}$$

We can stop there. Each column contains a pivot, there are no free variables, the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$, and therefore the columns are linearly independent.