Mathematics 210
Homework 3
Answers

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 7 & 4 & k \end{bmatrix}$. Find a value of k that makes the columns of A linearly dependent.

Answer: We proceed by row reduction:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 7 & 4 & k \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -9 \\ 0 & -10 & k-21 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & -10 & k-21 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & k+9 \end{bmatrix}$$

If $k + 9 = 0$, then the columns of the matrix are not linearly independent; therefore, the (only) answer is $k = -9$.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 7 & 4 & k \\ 11 & 3 & h \end{bmatrix}$. Find values of h and k that makes the columns of A linearly dependent.

Answer: Again, we use row reduction.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 7 & 4 & k \\ 11 & 3 & h \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -9 \\ 0 & -10 & k-21 \\ 0 & -19 & h-33 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & -10 & k-21 \\ 0 & -19 & h-33 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & k+9 \\ 0 & 0 & h+24 \end{bmatrix}$$

The columns are linearly dependent provided that $k + 9 = 0$ and $h + 24 = 0$, so the only values of h and k that make the columns linearly dependent are $k = -9$ and $h = -24$.

3. (*continued*) Find values of h and k that makes the columns of A linearly independent.
Answer: This question is a simple follow-up to the last one. As long as either $k \neq -9$ **or** $h \neq -24$, the columns will be linearly independent.

4. Let $A = \begin{bmatrix} 7 & 3 & 2 \\ 2 & 4 & 11 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Is there a vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$? If so, is that vector unique?
Answer: We use row reduction on the augmented matrix (and begin by interchanging the first and third rows to make the arithmetic a lot simpler):

$$\begin{bmatrix} 7 & 3 & 2 & 1 \\ 2 & 4 & 11 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 4 & 11 & 2 \\ 7 & 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 9 & -4 \\ 0 & -4 & -5 & -20 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 9 & -4 \\ 0 & 0 & 13 & -28 \end{bmatrix}$$

The form of this matrix tells us that there is a vector $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{b}$, and it is unique: each row of A has a pivot, as does each column.

5. Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Is there a vector $\mathbf{x}$ so that $A\mathbf{x} = \mathbf{b}$? If so, is that vector unique?

Answer: We know from the start that if there is such a vector, it cannot be unique, because it is impossible for each column of A to contain a pivot. We use row reduction to see if we end up with a pivot in the last column of the augmented matrix:

$$\begin{bmatrix} 3 & 4 & 5 & 2 \\ 6 & 7 & 8 & 3 \end{bmatrix} \to \begin{bmatrix} 3 & 4 & 5 & 2 \\ 0 & -1 & -2 & -1 \end{bmatrix}$$

We can now see that there are many solutions of the equation $A\mathbf{x} = \mathbf{b}$.

6. Suppose that the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent. Suppose that T is a linear transformation. Show that the set of vectors $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is linearly dependent.

Answer: Because the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, we know that we can write an equation $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$, where not all of the numbers a, b, and c are 0. Therefore, $T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = T(\mathbf{0}) = \mathbf{0}$. On the other hand, $T(a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w})$, because T is a linear transformation. Combining all of these equations, we get the equation $aT(\mathbf{u}) + bT(\mathbf{v}) + cT(\mathbf{w}) = \mathbf{0}$. Because we know that not all of a, b, and c are 0, this equation shows that the set $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ is linearly dependent.

7. Find a set of linearly independent vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ and a linear transformation T so that the vectors $\{T(\mathbf{u}), T(\mathbf{v}), T(\mathbf{w})\}$ are linearly dependent.

Answer: There are many possible answers here. The set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent. Let $T(\mathbf{v}) = \mathbf{0}$. Then the set $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ contains $\mathbf{0}$, and we discussed in class why any set of vectors containing $\mathbf{0}$ is automatically linearly dependent.

8. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation defined by the two formulas $T(\mathbf{e}_1) = \mathbf{e}_1$ and $T(\mathbf{e}_2) = 2\mathbf{e}_1 - 2\mathbf{e}_2$. What is the standard matrix of the linear transformation T?

Answer: The recipe to find the standard matrix is just to put the vectors $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ into a matrix, so the answer is $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 3 & 4 \end{bmatrix}$. Define a linear transformation $A : \mathbf{R}^3 \to \mathbf{R}^2$ with the formula $T(\mathbf{x}) = A\mathbf{x}$. Is T onto?

Answer: We apply row reduction: $\begin{bmatrix} 4 & 6 & 8 \\ 2 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 4 & 6 & 8 \\ 0 & 0 & 0 \end{bmatrix}$. Because the last row does not contain a pivot, T is not surjective.

10. (*continued*) Is T one-to-one?

Answer: Because the last two columns do not contains pivots, T is not injective.