1. Let \( A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \), \( B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \), and \( C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} \). Verify that \( AB = AC \), even though \( B \neq C \).

**Answer:** We have
\[
\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}.
\]

2. Suppose that \( A, B, \) and \( C \) are all \( n \times n \) matrices. Suppose also that \( AB = AC \), but \( B \neq C \). Prove that \( A \) is not invertible.

**Answer:** Suppose that \( AB = AC \) and \( A \) is invertible. Then multiplying both sides of \( AB = AC \) on the left by \( A^{-1} \), we have
\[
A^{-1}(AB) = A^{-1}(AC) \implies (A^{-1}A)B = (A^{-1}A)C \implies B = C.
\]
This is a contradiction, so we can conclude that \( A \) is not invertible.

3. Find a \( 2 \times 3 \) matrix \( A \) and a \( 3 \times 2 \) matrix \( B \) so that \( AB = I_2 \). Verify that \( BA \neq I_3 \).

**Answer:** Let \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \). Then \( AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \), but \( BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

4. Suppose that \( A \) is an \( n \times m \) matrix and \( B \) an \( m \times n \) matrix, and that \( AB = I \). Let \( T_A : \mathbb{R}^m \to \mathbb{R}^n \) be defined by \( T_A(x) = Ax \). Show that \( T_A \) is a surjective linear transformation by showing that the equation \( Ax = b \) can be solved for any vector \( b \).

**Answer:** Given any vector \( b \), we see that \( T_A(Bb) = A(Bb) = (AB)b = Ib = b \), so the equation \( T_A(x) = b \) is always solvable.

5. (continued) Let \( T_B : \mathbb{R}^n \to \mathbb{R}^m \) be defined by \( T_B(y) = By \). Show that \( T_B \) is an injective linear transformation by showing that the equation \( By = 0 \) has only the trivial solution \( y = 0 \).

**Answer:** Suppose that \( By = 0 \). Multiply both sides of the equation by \( A \):
\[
By = 0 \\
A(By) = A0 = 0 \\
(AB)y = 0 \\
y = 0
\]

6. Let \( A = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 3 & -1 \\ 8 & 9 & 2 \end{bmatrix} \).

Use row reduction to compute \( A^{-1} \).

**Answer:** We write \( I_3 \) next to the matrix \( A \), and row reduce.
\[
\begin{bmatrix} 2 & 0 & 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 8 & 9 & 2 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \\ -8 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.
\]
9. Let \( A = \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 3 & -7 & -2 & 1 & 0 \\ 0 & 9 & -10 & -4 & 0 & 1 \end{bmatrix} \). This means that \( A \). Those two equations amount to the equation \( \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 11 & 2 & -3 & 1 \end{bmatrix} \). We now set \( \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 11 \end{bmatrix} \). Therefore, \( A^{-1} = \begin{bmatrix} -\frac{9}{11} & \frac{9}{11} & -\frac{3}{11} \\ -\frac{10}{11} & \frac{10}{11} & \frac{7}{11} \\ -\frac{2}{11} & \frac{2}{11} & -\frac{3}{11} \end{bmatrix} \).

7. Suppose that \( A \) is a \( 2 \times 2 \) matrix, that \( A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 20 \end{bmatrix} \), and that \( A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 22 \\ 22 \end{bmatrix} \). Find the matrix \( A \).

Answer: Those two equations amount to the equation \( A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 20 \end{bmatrix} \). This means that \( A = \begin{bmatrix} 17 & 22 \\ 20 & 22 \end{bmatrix} \). Therefore, \( A = \begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix} \).

8. Suppose that \( A \) and \( B \) are \( 2 \times 2 \) matrices, that \( AB = \begin{bmatrix} 2 & 11 \\ 4 & 5 \end{bmatrix} \), and that \( B = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \). Find \( A \).

Answer: We have \( A = (AB)B^{-1} = \begin{bmatrix} 2 & 11 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -41 & 25 \\ 3 & -1 \end{bmatrix} \).

9. Let \( A = \begin{bmatrix} 3 & 4 & 5 & 2 \\ 1 & 3 & 2 & 3 \\ 3 & 2 & 1 & 9 \end{bmatrix} \) and let \( B = \begin{bmatrix} 6 & 3 \\ 2 & 4 \\ 8 & 3 \\ 1 & 7 \end{bmatrix} \). What is \( AB \)?

Answer: We have \( AB = \begin{bmatrix} 68 & 54 \\ 31 & 42 \\ 39 & 83 \end{bmatrix} \).

10. Find a \( 2 \times 2 \) matrix \( A \), which does not contain the number 0 in any of its 4 positions, so that \( A^2 \) is a matrix containing 4 zeroes.

Answer: If \( A^2 \) contains only zeroes, then \( A^2 \) is not invertible, and therefore \( A \) cannot be invertible. If \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), this means that \( ad - bc = 0 \), or \( ad = bc \). We can try setting \( a = 1 \) and seeing what happens.

This means that \( A = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix} \). We compute that \( A^2 = \begin{bmatrix} 1 + bc & b + b^2c \\ c + bc^2 & bc + b^2c^2 \end{bmatrix} = \begin{bmatrix} 1 + bc & b(1 + bc) \\ c(1 + bc) & bc(1 + bc) \end{bmatrix} \).

We now set \( bc = -1 \) (so that \( 1 + bc = 0 \)), and miraculously this equation makes all 4 of the matrix entries equal to 0. So we can try taking \( b = 1 \) and \( c = -1 \), and setting \( A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \). Sure enough, \( A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).