Mathematics 210
Homework 4
Answers

1. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$. Verify that $AB = AC$, even though $B \neq C$.

Answer: We have $\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$.

2. Suppose that A, B, and C are all $n \times n$ matrices. Suppose also that $AB = AC$, but $B \neq C$. Prove that A is not invertible.

Answer: Suppose that $AB = AC$ and A is invertible. Then multiplying both sides of $AB = AC$ on the left by A^{-1}, we have

$$\begin{aligned} AB &= AC \\ A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ B &= C \end{aligned}$$

This is a contradiction, so we can conclude that A is not invertible.

3. Find a 2×3 matrix A and a 3×2 matrix B so that $AB = I_2$. Verify that $BA \neq I_3$.

Answer: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but $BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4. Suppose that A is an $n \times m$ matrix and B an $m \times n$ matrix, and that $AB = I$. Let $T_A : \mathbf{R}^m \to \mathbf{R}^n$ be defined by $T_A(\mathbf{x}) = A\mathbf{x}$. Show that T_A is a surjective linear transformation by showing that the equation $A\mathbf{x} = \mathbf{b}$ can be solved for any vector $\mathbf{b}$.

Answer: Given any vector $\mathbf{b}$, we see that $T_A(B\mathbf{b}) = A(B\mathbf{b}) = (AB)\mathbf{b} = I\mathbf{b} = \mathbf{b}$, so the equation $T_A(\mathbf{x}) = \mathbf{b}$ is always solvable.

5. (*continued*) Let $T_B : \mathbf{R}^n \to \mathbf{R}^m$ be defined by $T_B(\mathbf{y}) = B\mathbf{y}$. Show that T_B is an injective linear transformation by showing that the equation $B\mathbf{y} = \mathbf{0}$ has only the trivial solution $\mathbf{y} = \mathbf{0}$.

Answer: Suppose that $B\mathbf{y} = \mathbf{0}$. Multiply both sides of the equation by A:

$$\begin{aligned} B\mathbf{y} &= \mathbf{0} \\ A(B\mathbf{y}) &= A\mathbf{0} = \mathbf{0} \\ (AB)\mathbf{y} &= \mathbf{0} \\ \mathbf{y} &= \mathbf{0} \end{aligned}$$

6. Let

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 3 & -1 \\ 8 & 9 & 2 \end{bmatrix}.$$

Use row reduction to compute A^{-1}.

Answer: We write I_3 next to the matrix A, and row reduce.

$$\begin{bmatrix} 2 & 0 & 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 8 & 9 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 8 & 9 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -8 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -7 & -2 & 1 & 0 \\ 0 & 9 & -10 & -4 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{7}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 9 & -10 & -4 & 0 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{7}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 11 & 2 & -3 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{11} \end{bmatrix}} \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{7}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{11} & -\frac{3}{11} & \frac{1}{11} \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{7}{3} \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{22} & \frac{9}{22} & -\frac{3}{22} \\ 0 & 1 & 0 & -\frac{8}{33} & -\frac{10}{33} & \frac{7}{33} \\ 0 & 0 & 1 & \frac{2}{11} & -\frac{3}{11} & \frac{1}{11} \end{bmatrix}$$

Therefore, $A^{-1} = \begin{bmatrix} \frac{5}{22} & \frac{9}{22} & -\frac{3}{22} \\ -\frac{8}{33} & -\frac{10}{33} & \frac{7}{33} \\ \frac{2}{11} & -\frac{3}{11} & \frac{1}{11} \end{bmatrix}$.

7. Suppose that A is a 2×2 matrix, that $A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 20 \end{bmatrix}$, and that $A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 22 \\ 22 \end{bmatrix}$. Find the matrix A.

Answer: Those two equations amount to the equation $A \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 22 \\ 20 & 22 \end{bmatrix}$. This means that $A = \begin{bmatrix} 17 & 22 \\ 20 & 22 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 17 & 22 \\ 20 & 22 \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 2 & -3 \\ -1 & 3 \end{bmatrix} = \left(\frac{1}{3}\right) \begin{bmatrix} 17 & 22 \\ 20 & 22 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 3 \end{bmatrix} = \left(\frac{1}{3}\right) \begin{bmatrix} 12 & 15 \\ 18 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$. Therefore, $A = \begin{bmatrix} 4 & 5 \\ 6 & 2 \end{bmatrix}$.

8. Suppose that A and B are 2×2 matrices, that $AB = \begin{bmatrix} 2 & 11 \\ 4 & 5 \end{bmatrix}$, and that $B = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$. Find A.

Answer: We have $A = (AB)B^{-1} = \begin{bmatrix} 2 & 11 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -41 & 25 \\ 3 & -1 \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 3 & 4 & 5 & 2 \\ 1 & 3 & 2 & 3 \\ 3 & 2 & 1 & 9 \end{bmatrix}$ and let $B = \begin{bmatrix} 6 & 3 \\ 2 & 4 \\ 8 & 3 \\ 1 & 7 \end{bmatrix}$. What is AB?

Answer: We have $AB = \begin{bmatrix} 68 & 54 \\ 31 & 42 \\ 39 & 83 \end{bmatrix}$.

10. Find a 2×2 matrix A, which does not contain the number 0 in any of its 4 positions, so that A^2 is a matrix containing 4 zeroes.

Answer: If A^2 contains only zeroes, then A^2 is not invertible, and therefore A cannot be invertible. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, this means that $ad - bc = 0$, or $ad = bc$. We can try setting $a = 1$ and seeing what happens. This means that $A = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}$. We compute that $A^2 = \begin{bmatrix} 1 + bc & b + b^2c \\ c + bc^2 & bc + b^2c^2 \end{bmatrix} = \begin{bmatrix} 1 + bc & b(1 + bc) \\ c(1 + bc) & bc(1 + bc) \end{bmatrix}$. We now set $bc = -1$ (so that $1 + bc = 0$), and miraculously this equation makes all 4 of the matrix entries equal to 0. So we can try taking $b = 1$ and $c = -1$, and setting $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Sure enough, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.