Mathematics 210
Homework 5
Answers

1. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 8 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$. Let $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$. Determine whether or not $\mathbf{p}$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Answer: We could row-reduce:

$$\begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 8 & 0 & 0 \\ 6 & 6 & 6 & 6 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}} \begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & -3 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

but it's just as simple to notice that $\mathbf{p} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3$, and conclude that $\mathbf{p}$ is indeed in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

2. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 8 \\ 6 \\ 4 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 6 \end{bmatrix}$. Let $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 6 \\ 2 \end{bmatrix}$. Determine whether or not $\mathbf{p}$ is in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Answer: Again, we row-reduce:

$$\begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 8 & 0 & 0 \\ 6 & 6 & 3 & 6 \\ 9 & 4 & 6 & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & -5 & -6 & -10 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{5}{8} & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & -6 & -10 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{6}{5} & 1 \end{bmatrix}} \begin{bmatrix} 3 & 3 & 4 & 4 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & -5 & -2 \\ 0 & 0 & 0 & -\frac{38}{5} \end{bmatrix}$$

The pivot in the last column shows that $\mathbf{p}$ is not in the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

3. Let $A = \begin{bmatrix} 3 & 4 & 5 & -1 & 2 \\ 2 & 3 & 2 & 3 & 2 \\ 7 & 6 & 6 & 5 & 4 \end{bmatrix}$. Find a basis for $\operatorname{Col} A$.

Answer: We begin our row-reduction:

$$\begin{bmatrix} 3 & 4 & 5 & -1 & 2 \\ 2 & 3 & 2 & 3 & 2 \\ 7 & 6 & 6 & 5 & 4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{7}{3} & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 4 & 5 & -1 & 2 \\ 0 & \frac{1}{3} & -\frac{4}{3} & \frac{11}{3} & \frac{2}{3} \\ 0 & -\frac{10}{3} & -\frac{17}{3} & \frac{22}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}} \begin{bmatrix} 3 & 4 & 5 & -1 & 2 \\ 0 & 1 & -4 & 11 & 2 \\ 0 & -10 & -17 & 22 & -2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & 4 & 5 & -1 & 2 \\ 0 & 1 & -4 & 11 & 2 \\ 0 & 0 & -57 & 132 & 18 \end{bmatrix}$$

This is sufficient for us to see that there is a pivot in each of the first three columns, and therefore $\begin{bmatrix}3\\2\\7\end{bmatrix}$, $\begin{bmatrix}4\\3\\6\end{bmatrix}$, and $\begin{bmatrix}5\\2\\6\end{bmatrix}$ is a basis for $\operatorname{Col} A$.

4. (*Continued*) Find a basis for $\operatorname{Nul} A$.

Answer: We continue row-reduction to reduced row-echelon form:

$$\begin{bmatrix}3 & 4 & 5 & -1 & 2\\0 & 1 & -4 & 11 & 2\\0 & 0 & -57 & 132 & 18\end{bmatrix} \xrightarrow{\begin{bmatrix}1&0&0\\0&1&0\\0&0&-\frac{1}{57}\end{bmatrix}} \begin{bmatrix}3 & 4 & 5 & -1 & 2\\0 & 1 & -4 & 11 & 2\\0 & 0 & 1 & -\frac{44}{19} & -\frac{6}{19}\end{bmatrix} \xrightarrow{\begin{bmatrix}1&0&-5\\0&1&4\\0&0&1\end{bmatrix}} \begin{bmatrix}3 & 4 & 0 & \frac{201}{19} & \frac{68}{19}\\0 & 1 & 0 & \frac{33}{19} & \frac{14}{19}\\0 & 0 & 1 & -\frac{44}{19} & -\frac{6}{19}\end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix}1&-4&0\\0&1&0\\0&0&1\end{bmatrix}} \begin{bmatrix}3 & 0 & 0 & \frac{69}{19} & \frac{12}{19}\\0 & 1 & 0 & \frac{33}{19} & \frac{14}{19}\\0 & 0 & 1 & -\frac{44}{19} & -\frac{6}{19}\end{bmatrix} \xrightarrow{\begin{bmatrix}\frac{1}{3}&0&0\\0&1&0\\0&0&1\end{bmatrix}} \begin{bmatrix}1 & 0 & 0 & \frac{23}{19} & \frac{4}{19}\\0 & 1 & 0 & \frac{33}{19} & \frac{14}{19}\\0 & 0 & 1 & -\frac{44}{19} & -\frac{6}{19}\end{bmatrix}$$

This tells us that

$$\begin{aligned}x_1 + \frac{23}{19}x_4 + \frac{4}{19}x_5 &= 0\\ x_2 + \frac{33}{19}x_4 + \frac{14}{19}x_5 &= 0\\ x_3 - \frac{44}{19}x_4 - \frac{6}{19}x_5 &= 0\end{aligned}$$

Therefore, a vector in $\operatorname{Nul} A$ satisfies $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}-\frac{23}{19}x_4\\-\frac{33}{19}x_4\\\frac{44}{19}x_4\\x_4\\0\end{bmatrix} + \begin{bmatrix}-\frac{4}{19}x_5\\-\frac{14}{19}x_5\\\frac{6}{19}x_5\\0\\x_5\end{bmatrix} = x_4\begin{bmatrix}-\frac{23}{19}\\-\frac{33}{19}\\\frac{44}{19}\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-\frac{4}{19}\\-\frac{14}{19}\\\frac{6}{19}\\0\\1\end{bmatrix}$, so one possible basis for $\operatorname{Nul} A$ is $\begin{bmatrix}-\frac{23}{19}\\-\frac{33}{19}\\\frac{44}{19}\\1\\0\end{bmatrix}$, $\begin{bmatrix}-\frac{4}{19}\\-\frac{14}{19}\\\frac{6}{19}\\0\\1\end{bmatrix}$. It is worth checking that these are in $\operatorname{Nul} A$ by multiplying both vectors by A and verifying that the result is $\begin{bmatrix}0\\0\\0\end{bmatrix}$.

5. Find a basis for the subspace spanned by $\begin{bmatrix}1\\-1\\-2\\5\end{bmatrix}$, $\begin{bmatrix}2\\-3\\-1\\6\end{bmatrix}$, $\begin{bmatrix}0\\2\\-6\\8\end{bmatrix}$, $\begin{bmatrix}-1\\4\\-7\\7\end{bmatrix}$, and $\begin{bmatrix}3\\-8\\9\\5\end{bmatrix}$.

Answer: Again, we row-reduce:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & 5 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -10 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 1 & -2 & -3 & 5 \\ 0 & 1 & -2 & -3 & 5 \\ 0 & 1 & -2 & -3 & \frac{5}{2} \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 1 & -2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix}$$

Because the pivots are in columns 1, 2, and 5, we know that a basis is given by $\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ -8 \\ 9 \\ 5 \end{bmatrix}$.

6. Suppose that A is a 2×3 and B is a 3×2 matrix. Prove that BA cannot be invertible.

Answer: Because A is a 2×3 matrix, we know that $\dim \operatorname{Nul} A + \operatorname{rank} A = 3$. Moreover, $\operatorname{rank} A \leq 2$, because A has only 2 rows. Therefore, $\dim \operatorname{Nul} A$ must be at least 1, so there must be a non-zero vector $\mathbf{x} \in \operatorname{Nul} A$. This means that $\mathbf{x} \neq \mathbf{0}$, while $A\mathbf{x} = \mathbf{0}$.

Suppose that BA is invertible. Then there is a matrix C so that $C(BA) = I$. Then on the one hand, $C(BA)\mathbf{x} = I\mathbf{x} \neq \mathbf{0}$, while on the other hand $C(BA)\mathbf{x} = CB(A\mathbf{x}) = CB\mathbf{0} = \mathbf{0}$. This is a contradiction, so we know that BA cannot be invertible.

7. Compute $\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & 4 & 1 & 0 \\ 8 & 5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix}$.

Answer: We have

$$\begin{vmatrix} 6 & 3 & 2 & 4 & 0 \\ 9 & 0 & 4 & 1 & 0 \\ 8 & 5 & 6 & 7 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 2 & 3 & 2 & 0 \end{vmatrix} = (-3) \begin{vmatrix} 3 & 2 & 4 & 0 \\ 0 & 4 & 1 & 0 \\ 5 & 6 & 7 & 1 \\ 2 & 3 & 2 & 0 \end{vmatrix} = (-3)(-1) \begin{vmatrix} 3 & 2 & 4 \\ 0 & 4 & 1 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= 3 \left(3 \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} + 2 \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} \right) = 3(3 \cdot 5 + 2 \cdot (-14)) = -39$$

8. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and let k be any real number. Find a formula relating $\det(kA)$ to $\det A$.

Answer: A bit of computation shows that $\det(kA) = k^2 \det A$.

9. Find values of h and k so that $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 3 \\ h \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ 2 \\ k \end{bmatrix}$.

Answer: Again, we row-reduce:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ h & 2 & 4 \\ 4 & k & 5 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -h & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 2-h & 4-2h \\ 0 & k-4 & -3 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & h-2 & 1 & 0 \\ 0 & 4-k & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 10-5h \\ 0 & 0 & 3k-15 \end{bmatrix}$$

This is not a contradictory system of equations only if $10 - 5h = 0$ and $3k - 15 = 0$, so the only possible values are $h = 2$ and $k = 5$.

10. Find a 2×2 matrix A and a non-zero vector $\mathbf{v} \in \mathbf{R}^2$ so that $\mathbf{v}$ is an element of both $\operatorname{Col} A$ and $\operatorname{Nul} A$.

Answer: One possible answer is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.