Mathematics 210
Homework 6
Answers

1. Suppose that A and B are n-by-n matrices, and A is singular. Show that both AB and BA must also be singular.

Answer: Because A is singular, we know that $\det A = 0$. Then we can compute that $\det(AB) = (\det A)(\det B) = 0$, and therefore AB must also be singular. Similarly, $\det(BA) = (\det B)(\det A) = 0$.

2. Let $P = (a, b)$ and $Q = (c, d)$ be 2 points in the xy-plane. Show that the equation

$$\begin{vmatrix} 1 & 1 & 1 \\ x & a & c \\ y & b & d \end{vmatrix} = 0$$

gives an equation for the line passing through P and Q.

Answer: Expanding the determinant along the first column yields the equation $\begin{vmatrix} a & c \\ b & d \end{vmatrix} - x\begin{vmatrix} 1 & 1 \\ b & d \end{vmatrix} + y\begin{vmatrix} 1 & 1 \\ a & c \end{vmatrix} = 0$, which is indeed a linear equation in x and y, so we now know that this is indeed the equation for a line.

We could proceed with horrible algebra to verify that this line really does pass through the points P and Q, but there is a more elegant solution. Substitute $x = a$ and $y = b$ into the determinant, and we get $\begin{vmatrix} 1 & 1 & 1 \\ a & a & c \\ b & b & d \end{vmatrix}$, which we know is 0 because the first and second columns are both 0. Similarly, substitute $x = c$ and $y = d$ into the determinant, and we get a determinant with equal first and third columns, which again must be 0.

3. Let

$$T = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Use row operations to show that $T = (b - a)(c - a)(c - b)$.

Answer: We have

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} = (b-a)(c-a)(c-b).$$

4. If A is an invertible matrix, show that

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Answer: We know that $AA^{-1} = I$, and therefore $\det(AA^{-1}) = \det I = 1$. We also know that in general, $\det(AB) = \det A \det B$, and applying that formula with $B = A^{-1}$ yields $\det A \det(A^{-1}) = 1$. Solving, we get $\det(A^{-1}) = 1/\det A$.

5. Let V be a vector space, and $\mathbf{v} \in V$. Show using only the definition of a vector space that $0\mathbf{v} = \mathbf{0}$.

Answer: Use the vector space property that $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$, with $a = b = 0$, to give the equation $0\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$. Now add $-0\mathbf{v}$ to both sides of the equation, and the result is $0\mathbf{v} = \mathbf{0}$.

6. Let V be a vector space, $\mathbf{0} \in V$, and let k be any real number. Show using only the definition of a vector space that $k\mathbf{0} = \mathbf{0}$.

Answer: This is almost the same as the last problem. We know that $\mathbf{v} + \mathbf{0} = \mathbf{v}$. Apply this equation with $\mathbf{v} = \mathbf{0}$, and we get $\mathbf{0}+\mathbf{0} = \mathbf{0}$. Now multiply both sides by k, and we have $k(\mathbf{0}+\mathbf{0}) = k\mathbf{0}$. One of the properties of a vector space is that $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$. We use that property to conclude that $k\mathbf{0} + k\mathbf{0} = k\mathbf{0}$. Now we add $-(k\mathbf{0})$ to both sides of the equation, and the result is $k\mathbf{0} = \mathbf{0}$.

7. Let $H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3 : x + y - z = 0 \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

Answer: Notice that $z = x + y$, so $H = \left\{ \begin{bmatrix} x \\ y \\ x+y \end{bmatrix} \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.
Because we showed in class that the span of a set of vectors is a subspace, we are done.

8. Let $H = \{p(x) \in \mathbf{P} : p(1) = 0\}$. Decide whether or not H is a subspace of $\mathbf{P}$. Be sure to explain your reasoning fully.

Answer: The set H is indeed a subspace of $\mathbf{P}$. First, the 0-polynomial is an element of H. Second, suppose that $p(x), q(x) \in H$. Then we know that $p(1) = q(1) = 0$, and therefore $(p + q)(1) = 0 + 0 = 0$, so $p(x) + q(x) \in H$. Finally, if $p(x) \in H$ and r is a scalar, then $rp(1) = r \cdot 0 = 0$, so $rp(x) \in H$.

9. Suppose that H and K are subspaces of a vector space V. Define

$$H + K = \{\mathbf{w} \in V : \mathbf{w} = \mathbf{h} + \mathbf{k},\ \mathbf{h} \in H,\ \mathbf{k} \in K\}.$$

Show that $H + K$ is a subspace of V.

Answer: First, we know that $\mathbf{0} \in H$ and $\mathbf{0} \in K$, so $\mathbf{0} + \mathbf{0} \in H + K$. Therefore, $\mathbf{0}$ is an element of $H + K$.

Second, suppose that $\mathbf{w}_1, \mathbf{w}_2 \in H + K$. Then $\mathbf{w}_1 = \mathbf{h}_1 + \mathbf{k}_1$ and $\mathbf{w}_2 = \mathbf{h}_2 + \mathbf{k}_2$, where $\mathbf{h}_1, \mathbf{h}_2 \in H$ and $\mathbf{k}_1, \mathbf{k}_2 \in K$. We compute that $\mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{h}_1 + \mathbf{h}_2) + (\mathbf{k}_1 + \mathbf{k}_2)$. Because $\mathbf{h}_1 + \mathbf{h}_2 \in H$ and $\mathbf{k}_1 + \mathbf{k}_2 \in K$, we know that $\mathbf{w}_1 + \mathbf{w}_2 \in H + K$.

Finally, suppose that $\mathbf{w} \in H + K$ and r is a scalar. Let $\mathbf{w} = \mathbf{h} + \mathbf{k}$, where $\mathbf{h} \in H$ and $\mathbf{k} \in K$. Then $r\mathbf{w} = r(\mathbf{h} + \mathbf{k}) = r\mathbf{h} + r\mathbf{k}$. Because H is a subspace, we know that $r\mathbf{h} \in H$; similarly, $r\mathbf{k} \in K$. Therefore $r\mathbf{w} \in H + K$.

10. Let $H = \left\{ \begin{bmatrix} c - 4d \\ 2c + d \\ c - d \end{bmatrix} \in \mathbf{R}^3 : c, d \in \mathbf{R} \right\}$. Show that H is a subspace of $\mathbf{R}^3$ by finding vectors $\mathbf{u}$ and $\mathbf{v}$ so that H is the span of $\{\mathbf{u}, \mathbf{v}\}$.

Answer: We have $H = \left\{ \begin{bmatrix} c - 4d \\ 2c + d \\ c - d \end{bmatrix} \right\} = \left\{ c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \right\}$.