Mathematics 210
Homework 7
Answers

1. Suppose that $H = \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find a basis for H.

Answer: Because the two vectors $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent, they are a basis for H.

2. (*Continued*) Find a 4-by-4 matrix A so that $\text{Col}\, A = H$.

Answer: There are many possible answers. One is $A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

3. (*Continued*) Find a 4-by-4 matrix B so that $\text{Nul}\, B = H$.

Answer: This is quite a bit harder than the previous two questions. Suppose that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is an element of $\text{Nul}\, B$. Then we know that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$. This amounts to the 4 equations

$$\begin{aligned} x_1 &= 3c + 2d \\ x_2 &= \phantom{3c + {}} 3d \\ x_3 &= c \\ x_4 &= d \end{aligned}$$

This amounts to the two equations

$$\begin{aligned} x_1 \qquad - 3x_3 - 2x_4 &= 0 \\ x_2 \qquad\qquad - 3x_4 &= 0 \end{aligned}$$

and so one possible answer is $B = \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Because B has rank 2, we know that its null space is 2 dimensional. We can then check that the two vectors which span H are in the nullspace, and that completes the problem.

4. Let $W_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2 : xy = 0 \right\}$. Is W_1 a subspace of $\mathbf{R}^2$? If so, prove your answer; if not, explain why not.

Answer: The set W_1 is *not* a subspace. The easiest way to see this is to notice that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are both in W_1, but the sum of those two vectors is not in W_1.

5. Let $W_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2 : x + y = 0 \right\}$. Is W_2 a subspace of $\mathbf{R}^2$? If so, prove your answer; if not, explain why not.

Answer: The set W_2 is indeed a subspace of $\mathbf{R}^2$. First, notice that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W_2$. Second, suppose that $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in W_2$. This means that $a + b = 0$ and $c + d = 0$. The sum of those two vectors, $\begin{bmatrix} a+c \\ b+d \end{bmatrix}$, is also in W_2, because $(a + c) + (b + d) = 0$. Finally, if $\begin{bmatrix} a \\ b \end{bmatrix} \in W_2$, and k is a scalar, then $ka + kb = 0$, so $k \begin{bmatrix} a \\ b \end{bmatrix} \in W_2$.

6. Recall that $\mathbf{P}_2 = \{at^2 + bt + c\}$, the vector space of all polynomials of degree no more than 2. Define a function $T_1 : \mathbf{P}_2 \to \mathbf{R}^2$ by $T_1(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$. Show that the function T_1 is a linear transformation.

Answer: Suppose that $p, q \in \mathbf{P}_2$. Then $T_1(p + q) = \begin{bmatrix} p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(0) \\ q(1) \end{bmatrix} = T_1(p) + T_1(q)$. Also, if k is a scalar, then $T_1(kp) = \begin{bmatrix} kp(0) \\ kp(1) \end{bmatrix} = k \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} = kT_1(p)$. These two facts show that T_1 is a linear transformation.

7. (*Continued*) Find a polynomial $f(t) \in \mathbf{P}_2$ so that the kernel of T_1 is Span$\{f(t)\}$.
Answer: If $f(t)$ is in the kernel, then $T_1(f) = \mathbf{0}$, meaning that $f(0) = f(1) = 0$. We also know that $f(t) = at^2 + bt + c$, so we can conclude that $c = 0$ and $a + b = 0$. So $f(t) = a(t^2 - t)$, and we can write $\ker(T_1) = \text{Span}\{t^2 - t\}$.

8. Now define a function $T_2 : \mathbf{P}_2 \to \mathbf{R}^2$ with the formula $T_2(p) = \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix}$. (As usual, $p'(0)$ means the derivative of $p(t)$ evaluated at $t = 0$.) Show that T_2 is a linear transformation.
Answer: Suppose that $p, q \in \mathbf{P}_2$. Now, we have $T_2(p + q) = \begin{bmatrix} p(0) + q(0) \\ p'(0) + q'(0) \end{bmatrix} = \begin{bmatrix} p(0) \\ p'(0) \end{bmatrix} + \begin{bmatrix} q(0) \\ q'(0) \end{bmatrix} = T_2(p) + T_2(q)$. And if k is a scalar, $T_2(kp) == \begin{bmatrix} kp(0) \\ kp'(0) \end{bmatrix} = kT_2(p)$. These two equations show that T_2 is a linear transformation.

9. (*Continued*) Find a polynomial $g(t) \in \mathbf{P}_2$ so that the kernel of T_2 is Span$\{g(t)\}$.
Answer: If $T_2(g) = \mathbf{0}$, then we have $g(0) = g'(0) = 0$. Therefore, $g(t) = at^2$, so $\ker(T_2) = \text{Span}\{t^2\}$.

10. Now define a function $T_3 : \mathbf{P}_2 \to \mathbf{R}^2$ by $T_3(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$. You may assume that T_3 is a linear transformation; the proof is similar to the proofs for the function T_1. Find polynomials $h_1(t), h_2(t) \in \mathbf{P}_2$ so that $\{h_1(t), h_2(t)\}$ is a basis for the kernel of T_3.
Answer: Suppose that $T_3(h) = 0$. Then $h(0) = 0$, so $h(t) = at^2 + bt$. Therefore, $\ker(T_3) = \text{Span}(t, t^2\}$.