Mathematics 210
Homework 8
Answers

1. Let $H_1 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbf{R}^4 : a+b+c=0, a-b+c=0, b+2c=0 \right\}$. Find a basis for H_1 and find the dimension of H_1.

Answer: The vector space H_1 is determined by the set of solutions to those 3 equations. We set up and row-reduce the matrix corresponding to those equations, including a column for the variable d even though it does not appear:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \overset{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \overset{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \overset{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

This is enough to show that $a=b=c=0$ is the only solution. The variable d is completely undetermined, so that $H_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} \in \mathbf{R}^4 \right\}$. Therefore, a basis for H_1 is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and the dimension of H_1 is 1.

2. Let $H_2 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbf{R}^3 : 3a+2b+c=0 \right\}$. Find a basis for H_2, and find the dimension of H_2.

Answer: Because $a = -\frac{2}{3}b - \frac{1}{3}c$, we can write $H_2 = \left\{ \begin{bmatrix} -\frac{2}{3}b - \frac{1}{3}c \\ b \\ c \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -\frac{2}{3}b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}c \\ 0 \\ c \end{bmatrix} \right\} = \left\{ b \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$. This shows that a basis for H_2 is $\left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$, and the dimension of H_2 is 2.

3. Let $\mathcal{B}_1 = \{1, 2t, 4t^2 - 2t, 8t^3 - 12t^2\}$ be a basis for $\mathbf{P}_3$. Find $[5t^2 - 3t]_{\mathcal{B}_1}$. (This means the coordinate vector of $5t^2 - 3t$ relative to the basis $\mathcal{B}_1$.)

Answer: We fortunately can do this with less effort than you might think. Write

$$\begin{aligned} \mathbf{b}_1 &= 1 \\ \mathbf{b}_2 &= 2t \\ \mathbf{b}_3 &= 4t^2 - 2t \\ \mathbf{b}_4 &= 8t^3 - 12t^2 \end{aligned}$$

Then we can see that $5t^2 - 3t = \frac{5}{4}\mathbf{b}_3 + c_2\mathbf{b}_2 + c_1\mathbf{b}_1$, and a bit more experimentation shows that $c_2 = -\frac{1}{4}$ and $c_1 = 0$. Therefore, $[5t^2 - 3t]_{\mathcal{B}_1} = \begin{bmatrix} 0 \\ -\frac{1}{4} \\ \frac{5}{4} \\ 0 \end{bmatrix}$. We'll be able to check this answer after we work out the answer to the next question.

4. Let $\mathcal{B}_1 = \{1, 2t, 4t^2 - 2t, 8t^3 - 12t^2\}$ be a basis for $\mathbf{P}_3$. Let $\mathcal{E} = \{1, t, t^2, t^3\}$. Find the matrices $\underset{\mathcal{B}_1 \leftarrow \mathcal{E}}{P}$ and $\underset{\mathcal{E} \leftarrow \mathcal{B}_1}{P}$.

Answer: What makes this problem manageable is that $\underset{\mathcal{E} \leftarrow \mathcal{B}_1}{P} = [[\mathbf{b}_1]_\mathcal{E} [\mathbf{b}_2]_\mathcal{E} [\mathbf{b}_3]_\mathcal{E} [\mathbf{b}_4]_\mathcal{E}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 4 & -12 \\ 0 & 0 & 0 & 8 \end{bmatrix}$.

Then $\underset{\mathcal{B}_1 \leftarrow \mathcal{E}}{P} = (\underset{\mathcal{E} \leftarrow \mathcal{B}_1}{P})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$. We can use this matrix to check our answer to the previous question.

We can easily compute that $[5t^2 - 3t]_\mathcal{E} = \begin{bmatrix} 0 \\ -3 \\ 5 \\ 0 \end{bmatrix}$, and therefore

$$[5t^2 - 3t]_{\mathcal{B}_1} = \underset{\mathcal{B}_1 \leftarrow \mathcal{E}}{P} [5t^2 - 3t]_\mathcal{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 0 \\ -3 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{4} \\ \frac{5}{4} \\ 0 \end{bmatrix}.$$

5. Suppose that $\mathbf{a}_1 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the matrices $\underset{\mathcal{B} \leftarrow \mathcal{A}}{P}$ and $\underset{\mathcal{A} \leftarrow \mathcal{B}}{P}$.

Answer: One way to approach this problem is to use the formula at the very end of Section 4.7. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis, and write

$$\underset{\mathcal{B} \leftarrow \mathcal{A}}{P} = \underset{\mathcal{B} \leftarrow \mathcal{E}}{P}\,\underset{\mathcal{E} \leftarrow \mathcal{A}}{P} = (\underset{\mathcal{E} \leftarrow \mathcal{B}}{P})^{-1} \underset{\mathcal{E} \leftarrow \mathcal{A}}{P} = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{5}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 7 & 2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

$$\underset{\mathcal{A} \leftarrow \mathcal{B}}{P} = (\underset{\mathcal{B} \leftarrow \mathcal{A}}{P})^{-1} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}.$$

6. The matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}$ has an eigenvalue $\lambda = 3$. Find a basis of the eigenspace for λ.

Answer: We row-reduce $A - 3I$:

$$A - 3I = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the eigenspace, then $x_1 + 2x_2 + 3x_3 = 0$, or $x_1 = -2x_2 - 3x_3$. An element of the eigenspace is therefore $\begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Therefore, a basis of the eigenspace is given by $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

7. Show that λ is an eigenvalue for A if and only if λ is an eigenvalue for A^T. *Hint:* Consider $A - \lambda I$ and $A^T - \lambda I$.

Answer: The characteristic equation for A is $\det(A - \lambda I) = 0$. The characteristic equation for A^T is $\det(A^T - \lambda I) = 0$. Because $I = I^T$, and $\det B = \det B^T$ for any matrix B, we have $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$. Because A and A^T have the same characteristic equation, they have the same eigenvalues.

8. Suppose that A is an n-by-n matrix so that every row adds up to the same number s. Show that s is an eigenvalue for A by finding an eigenvector for A.

Answer: If every row of A adds up to a number s, then $A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. This shows that $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue s.

9. Suppose that B is an n-by-n matrix so that every column adds up to the same number s. Combine the previous two problems to show that s is an eigenvalue for B.

Answer: Every row of B^T adds up to s (because the columns of B are the rows of B^T). By the previous problem B^T has s as an eigenvalue. The problem before that shows that B and B^T have the same eigenvalues, so together this shows that B has s as an eigenvalue.

10. Suppose that A is a matrix, and $A^2 = 0$. Show that the only possible eigenvalue for A is 0.

Answer: Suppose that $A\mathbf{v} = \lambda\mathbf{v}$. Then $A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda^2\mathbf{v}$. But we also know that $A^2 = 0$, so $A^2\mathbf{v} = \mathbf{0}$. Combined, this all says that $\lambda^2\mathbf{v} = \mathbf{0}$. Because we know that $\mathbf{v} \neq \mathbf{0}$, we can conclude that $\lambda^2 = 0$, which in turn means that $\lambda = 0$.