Mathematics 210
Homework 9
Answers

1. Let $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$. (*Note:* This problem is sufficiently intricate to count as a double problem. Also, all eigenvalues in this problem are whole numbers.)

Answer: We start by finding the characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} \\
&= (-9-\lambda)\begin{vmatrix} 3-\lambda & 4 \\ 8 & 7-\lambda \end{vmatrix} - 4\begin{vmatrix} -8 & 4 \\ -16 & 7-\lambda \end{vmatrix} + 4\begin{vmatrix} -8 & 3-\lambda \\ -16 & 8 \end{vmatrix} \\
&= -(9+\lambda)(\lambda^2 - 10\lambda - 11) - 4(8\lambda + 8) + 4(-16 - 16\lambda) \\
&= -\lambda^3 + \lambda^2 + 101\lambda + 99 - 32\lambda - 32 - 64\lambda - 64 \\
&= -\lambda^3 + \lambda^2 + 5\lambda + 3
\end{aligned}$$

We need to solve $\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$, which factors as $(\lambda - 3)(\lambda + 1)^2 = 0$.

First, we find an eigenvector associated to the eigenvalue $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \xrightarrow{\begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}} \begin{bmatrix} 3 & -1 & -1 \\ 2 & 0 & -1 \\ 4 & -2 & -1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & -1 & -1 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 3 & -1 & -1 \\ 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & -1 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 3 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

This yields the equations $3x_1 - \frac{3}{2}x_3 = 0$ and $x_2 - \frac{1}{2}x_3 = 0$. Choose $x_3 = 2$, and then we have $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as an eigenvector.

Next, we compute eigenvectors associated to the eigenvalue $\lambda = -1$:

$$A + I = \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \xrightarrow{\begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{8} \end{bmatrix}} \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This leads to the single equation $2x_1 - x_2 - x_3 = 0$. Taking $x_2 = 2$ and $x_3 = 0$, we get the eigenvector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Taking $x_2 = 0$ and $x_3 = 2$, we get the eigenvector $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

Finally, we set

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

and check that $PD = AP$ to avoid computing P^{-1}.

3. Let $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$. (*Note:* This problem is sufficiently intricate to count as a double problem. Also, all eigenvalues in this problem are whole numbers.)

Answer: We start by computing the characteristic polynomial in order to compute the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ -3 & -6 & -6-\lambda \end{vmatrix} \\
&= (-1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ -6 & -6-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -3 & -6-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 2-\lambda \\ -3 & -6 \end{vmatrix} \\
&= (-1-\lambda)(\lambda^2 + 4\lambda) - 2(-2\lambda - 6) + 2(-3\lambda - 6) \\
&= -\lambda^3 - 5\lambda^2 - 4\lambda + 4\lambda + 12 - 6\lambda - 12 = -\lambda^3 - 5\lambda^2 - 6\lambda
\end{aligned}$$

We factor, and see that $\lambda^3 + 5\lambda^2 + 6\lambda = \lambda(\lambda+2)(\lambda+3)$.

We start by computing an eigenvector associated with the eigenvalue $\lambda = 0$:

$$A - 0I = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{bmatrix} \xrightarrow{\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}} \begin{bmatrix} 1 & -2 & -2 \\ 1 & 1 & 1 \\ 1 & -2 & -2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This leads to the equations $x_1 = 0$ and $x_2 + x_3 = 0$, and so one possible eigenvector is $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Next, we compute the eigenvector corresponding to $\lambda = -2$:

$$A + 2I = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 2 \\ -3 & -6 & -4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is sufficient to see that $x_1 + 2x_2 = 0$ and $-2x_3 = 0$, so an eigenvector is given by $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Next, we compute an eigenvector associated to $\lambda = -3$:

$$A + 3I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 2 \\ -3 & -6 & -3 \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This is sufficient to conclude that $x_2 = 0$ and $x_1 + x_3 = 0$, so an eigenvector is given by $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Finally, we set

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

and check that $PD = AP$ to avoid computing P^{-1}.

5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$.

Answer: We start with the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 0 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = \lambda^2 - \lambda - 1.$$

We solve $\lambda^2 - \lambda - 1 = 0$, and find that $\lambda = \frac{1 \pm \sqrt{5}}{2}$. To simplify notation below, we set $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\phi' = \frac{1-\sqrt{5}}{2} \approx -0.618$. Remember that $\phi^2 = \phi + 1$ and similarly for ϕ'.

First, we find an eigenvector corresponding to the eigenvalue $\lambda = \phi$:

$$A - \phi I = \begin{bmatrix} -\phi & 1 \\ 1 & 1 - \phi \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 - \phi \\ -\phi & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ \phi & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 - \phi \\ 0 & 0 \end{bmatrix}$$

This leads to the equation $x_1 + (1 - \phi)x_2 = 0$, and taking $x_2 = 1$ leads to the eigenvector $\begin{bmatrix} \phi - 1 \\ 1 \end{bmatrix}$.

For the eigenvalue $\lambda = \phi'$, we have a similar calculation:

$$A - \phi' I = \begin{bmatrix} -\phi' & 1 \\ 1 & 1 - \phi' \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 - \phi' \\ -\phi' & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ \phi' & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 - \phi' \\ 0 & 0 \end{bmatrix}$$

which leads to the eigenvector $\begin{bmatrix} \phi' - 1 \\ 1 \end{bmatrix}$.

We set

$$D = \begin{bmatrix} \phi & 0 \\ 0 & \phi' \end{bmatrix} \qquad P = \begin{bmatrix} \phi - 1 & \phi' - 1 \\ 1 & 1 \end{bmatrix}$$

and check that $PD = AP$ to avoid computing P^{-1}.

6. Suppose that λ is an eigenvalue for an invertible matrix A. Show that λ^{-1} is an eigenvalue for A^{-1}.

Answer: Suppose that $A\mathbf{x} = \lambda\mathbf{x}$. Multiply both sides of the equation on the left by A^{-1}. The left-hand side of the equation becomes $A^{-1}(A\mathbf{x}) = (A^{-1}A)\mathbf{x} = \mathbf{x}$, while the right-hand side becomes $A^{-1}(\lambda\mathbf{x}) = \lambda(A^{-1}\mathbf{x})$. We therefore have the equation $\mathbf{x} = \lambda(A^{-1}\mathbf{x})$, and multiplying both sides by λ^{-1}, gives the equation $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which is exactly the equation that says that $\mathbf{x}$ is an eigenvector for A^{-1} with eigenvalue λ^{-1}.

7. Recall that a square matrix A is *similar* to a square matrix B if there is an invertible matrix K so that $A = KBK^{-1}$. Suppose that A is similar to B, and B is similar to C. Show that A is similar to C.

Answer: If A is similar to B, then we can write $A = KBK^{-1}$. If B is similar to C, then we can write $B = JCJ^{-1}$. (It is important here not to assume that we can use the same matrix K in this second equation, and use a different variable J to denote a different invertible matrix.) Substituting, we have

$A = K(JCJ^{-1})K^{-1} = (KJ)C(KJ)^{-1}$. If we let $KJ = L$, we have $A = LCL^{-1}$, showing that A is similar to C.

8. Show that the 0 matrix (that is, the matrix which contains only the number 0) is similar only to itself.

Answer: Suppose that $0 = KBK^{-1}$. Multiply on the left by K^{-1} and on the right by K, and we have $K^{-1}0K = B$. But $K^{-1}0K = 0K = 0$, so we have the equation $0 = B$. Therefore, the only matrix that 0 is similar to is itself.

9. Suppose that A is similar to B. Show that $\det A = \det B$.

Answer: We start with $A = KBK^{-1}$. Then, using the facts that $\det(CD) = (\det C)(\det D)$ and $\det(C^{-1}) = 1/\det C$, we have $\det(A) = \det(KBK^{-1}) = (\det K)(\det B)(\det(K^{-1})) = (\det K)(\det B)(1/\det K) = \det B$, thereby showing that $\det A = \det B$.

10. Suppose that A is an n-by-n matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Prove that $\det A = \lambda_1\lambda_2, \cdots \lambda_n$.

Answer: If A is an n-by-n matrix with n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we can write $A = PDP^{-1}$, where $D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$. The previous problem showed that $\det A = \det D$, and $\det D = \lambda_1\lambda_2 \cdots \lambda_n$.