Mathematics 210
Homework 10
Due Friday, December 5, 2 PM

Please note that this homework is due at 2 PM. No late homework can be accepted. You must turn in your answers by the start of class on Friday.

1. Suppose that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that A is *not* diagonalizable. In other words, show that you cannot write $A = PDP^{-1}$, where D is a diagonal matrix.

2. Suppose that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$, where t is any non-zero real number. Show that A and B are similar.

3. Suppose that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$.

4. Suppose that $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$.

5. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose that A has 2 unequal eigenvalues λ_1 and λ_2. Show that $a + d = \lambda_1 + \lambda_2$.

6. The *Lucas numbers* are the sequence $L_1 = 1$, $L_2 = 3$, $L_3 = 4, \ldots$, defined by $L_{n+1} = L_n + L_{n-1}$. Find a formula for the Lucas numbers similar to the formula for the Fibonacci numbers that we found in class.

7. Suppose that a predator-prey problem (akin to the one in Section 5.6 of our text) is modelled using the matrix $A = \begin{bmatrix} 0.4 & 0.3 \\ -p & 1.2 \end{bmatrix}$. Show that if the value of $p = 0.325$, the populations of both the predator and the prey will grow. Find the eventual ratio of predator to prey.

8. (*Continued*) Now suppose that $p = 0.5$ in the matrix in the previous problem. Show that both the predator and the prey will eventually perish.

9. Solve the differential equations

$$\begin{aligned} y_1' &= -2y_1 - 5y_2 \\ y_2' &= y_1 + 4y_2 \end{aligned}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

10. Solve the differential equations

$$\begin{aligned} y_1' &= 3y_1 - 2y_2 \\ y_2' &= 2y_1 + 3y_2 \end{aligned}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

Mathematics 210
Homework 10
Answers

1. Suppose that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that A is *not* diagonalizable. In other words, show that you cannot write $A = PDP^{-1}$, where D is a diagonal matrix.

Answer: Suppose that we can write $A = PDP^{-1}$, with $D = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Then on the one hand, $A^2 = PD^2P^{-1} = P\begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix}P^{-1}$, and on the other hand $A^2 = 0$. This yields $0 = PD^2P^{-1}$. Multiply on the left by P^{-1}, on the right by P, and simplify, and we get $D^2 = 0$. Because of the form of D^2, we can conclude that $a = d = 0$, and therefore $D = 0$. But then the equation $A = PDP^{-1}$ gives $A = 0$, which is a contradiction.

2. Suppose that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$, where t is any non-zero real number. Show that A and B are similar.

Answer: We must show that we can write $A = PBP^{-1}$; clearly, it is much simpler to write $AP = PB$, with P an invertible matrix. Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $ad - bc \neq 0$. Then we have $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$. This equation reduces to $\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & at \\ 0 & ct \end{bmatrix}$. If we set $c = 0$ and $d = at$, we have no other conditions to satisfy. In particular, we can set $a = 1$, $b = 0$, and $d = t$, and let $P = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$.

3. Suppose that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$.

Answer: We start with the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 6 = \lambda^2 - 5\lambda - 2.$$

We solve $\lambda^2 - 5\lambda - 2 = 0$, and find that $\lambda = \frac{5\pm\sqrt{33}}{2}$.

We begin with $\lambda = \frac{5+\sqrt{33}}{2}$:

$$A - \frac{5+\sqrt{33}}{2}I = \begin{bmatrix} 1 - \frac{5+\sqrt{33}}{2} & 2 \\ 3 & 4 - \frac{5+\sqrt{33}}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3-\sqrt{33}}{2} & 2 \\ 3 & \frac{3-\sqrt{33}}{2} \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & \frac{3-\sqrt{33}}{2} \\ \frac{-3-\sqrt{33}}{2} & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & \frac{3-\sqrt{33}}{6} \\ \frac{-3-\sqrt{33}}{2} & 2 \end{bmatrix}$$

Because we know that the matrix has rank 1, we can just use the first line to find an eigenvector. We have $x_1 + \frac{3-\sqrt{33}}{6}x_2 = 0$. Set $x_2 = 1$, and then $x_1 = \frac{\sqrt{33}-3}{6}$. So an eigenvector is given by $\begin{bmatrix} \frac{\sqrt{33}-3}{6} \\ 1 \end{bmatrix}$.

The computation with $\lambda = \frac{5-\sqrt{33}}{2}$ is remarkably similar:

$$A - \frac{5-\sqrt{33}}{2}I = \begin{bmatrix} 1 - \frac{5-\sqrt{33}}{2} & 2 \\ 3 & 4 - \frac{5-\sqrt{33}}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3+\sqrt{33}}{2} & 2 \\ 3 & \frac{3+\sqrt{33}}{2} \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 3 & \frac{3+\sqrt{33}}{2} \\ \frac{-3+\sqrt{33}}{2} & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & \frac{3+\sqrt{33}}{6} \\ \frac{-3+\sqrt{33}}{2} & 2 \end{bmatrix}$$

leading to $\begin{bmatrix} \frac{-\sqrt{33}-3}{6} \\ 1 \end{bmatrix}$ as an eigenvector.

Therefore, we can set

$$P = \begin{bmatrix} \frac{\sqrt{33}-3}{6} & \frac{-\sqrt{33}-3}{6} \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}$$

4. Suppose that $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$. Find an invertible matrix P and diagonal matrix D so that $A = PDP^{-1}$.

Answer: We start with the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 6 = \lambda^2 - 5\lambda + 10.$$

We solve $\lambda^2 - 5\lambda + 10 = 0$, and find that $\lambda = \frac{5 \pm \sqrt{-15}}{2}$.

We start with $\lambda = \frac{5+\sqrt{-15}}{2}$, and then we will get the eigenvector corresponding to $\overline{\lambda}$ by using complex conjugation.

We have

$$A - \frac{5+\sqrt{-15}}{2} I = \begin{bmatrix} 1 - \frac{5+\sqrt{-15}}{2} & 2 \\ -3 & 4 - \frac{5+\sqrt{-15}}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3-\sqrt{-15}}{2} & 2 \\ -3 & \frac{3-\sqrt{-15}}{2} \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} -3 & \frac{3-\sqrt{-15}}{2} \\ \frac{-3-\sqrt{-15}}{2} & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & \frac{-3+\sqrt{-15}}{6} \\ \frac{-3-\sqrt{-15}}{2} & 2 \end{bmatrix}$$

We can use the first line to get an eigenvector. Take the equation $x_1 + \frac{-3+\sqrt{-15}}{6} x_2 = 0$, and set $x_2 = 1$, and we have $x_1 = \frac{3-\sqrt{-15}}{6}$. Therefore, an eigenvector is $\begin{bmatrix} \frac{3-\sqrt{-15}}{6} \\ 1 \end{bmatrix}$ and a corresponding eigenvector for the eigenvalue $\frac{5-\sqrt{-15}}{2}$ is $\begin{bmatrix} \frac{3+\sqrt{-15}}{6} \\ 1 \end{bmatrix}$.

Therefore, we can set

$$P = \begin{bmatrix} \frac{3-\sqrt{-15}}{6} & \frac{3+\sqrt{-15}}{6} \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \frac{5+\sqrt{-15}}{2} & 0 \\ 0 & \frac{5-\sqrt{-15}}{2} \end{bmatrix}$$

5. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose that A has 2 unequal eigenvalues λ_1 and λ_2. Show that $a+d = \lambda_1 + \lambda_2$.

Answer: This one is easy. The characteristic polynomial for A is $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$. We know that the polynomial factors as $\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Expanding and comparing the coefficient of λ gives $a + d = \lambda_1 + \lambda_2$.

6. The *Lucas numbers* are the sequence $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $\ldots$, defined by $L_{n+1} = L_n + L_{n-1}$. Find a formula for the Lucas numbers similar to the formula for the Fibonacci numbers that we found in class.

Answer: Set $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and $\phi' = \frac{1-\sqrt{5}}{2} \approx -0.618$. We showed in class that $A = PDP^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ \phi & \phi' \end{bmatrix}$, $D = \begin{bmatrix} \phi & 0 \\ 0 & \phi' \end{bmatrix}$, and $P^{-1} = -\frac{1}{\sqrt{5}} \begin{bmatrix} \phi' & -1 \\ -\phi & 1 \end{bmatrix}$, and that $\phi\phi' = -1$.

In this case, we have

$$\begin{bmatrix} L_n \\ L_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} = -\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi & \phi' \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & (\phi')^n \end{bmatrix} \begin{bmatrix} \phi' & -1 \\ -\phi & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$= -\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi & \phi' \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & (\phi')^n \end{bmatrix} \begin{bmatrix} 2\phi' - 1 \\ -2\phi + 1 \end{bmatrix}$$
$$= -\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \phi & \phi' \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & (\phi')^n \end{bmatrix} \begin{bmatrix} -\sqrt{5} \\ -\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \phi & \phi' \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & (\phi')^n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ \phi & \phi' \end{bmatrix} \begin{bmatrix} \phi^n \\ (\phi')^n \end{bmatrix} = \begin{bmatrix} \phi^n + (\phi')^n \\ \phi^{n+1} + (\phi')^{n+1} \end{bmatrix}$$

Therefore, $L_n = \phi^n + (\phi')^n$. As a quick check, we compute $\phi^{10} + (\phi')^{10} \approx 123$, and verify that this is correct.

7. Suppose that a predator-prey problem (akin to the one in Section 5.6 of our text) is modelled using the matrix $A = \begin{bmatrix} 0.4 & 0.3 \\ -p & 1.2 \end{bmatrix}$. Show that if the value of $p = 0.325$, the populations of both the predator and the prey will grow. Find the eventual ratio of predator to prey.

Answer: The characteristic polynomial is

$$\det A - \lambda I = \begin{vmatrix} 0.4 - \lambda & 0.3 \\ -0.325 & 1.2 - \lambda \end{vmatrix} = (0.4 - \lambda)(1.2 - \lambda) + 0.0975 = \lambda^2 - 1.6\lambda + 0.5775.$$

We apply the quadratic formula, and find that $\lambda = \frac{1.6 \pm \sqrt{0.25}}{2} = \frac{1.6 \pm 0.5}{2}$. The two roots are 1.05 and 0.55.

We care only about the larger eigenvalue, because that is the one which will dominate. The corresponding eigenvector is computed as follows:

$$A - 1.05I = \begin{bmatrix} -0.65 & 0.3 \\ -0.325 & 0.15 \end{bmatrix} \to \begin{bmatrix} -0.65 & 0.3 \\ 0 & 0 \end{bmatrix}$$

In the long term, the predator-prey ratio will be $0.3 : 0.65$, or $6 : 13$.

8. (*Continued*) Now suppose that $p = 0.5$ in the matrix in the previous problem. Show that both the predator and the prey will eventually perish.

Answer: Now, the characteristic polynomial is

$$\det A - \lambda I = \begin{vmatrix} 0.4 - \lambda & 0.3 \\ -0.5 & 1.2 - \lambda \end{vmatrix} = (0.4 - \lambda)(1.2 - \lambda) + 0.15 = \lambda^2 - 1.6\lambda + 0.63.$$

The quadratic formula yields $\lambda = \frac{1.6 \pm \sqrt{0.04}}{2} = \frac{1.6 \pm 0.2}{2}$. The two eigenvalues are 0.9 and 0.7. Because both are less than 1, in the long term, the population of both the predator and the prey will tend to 0.

9. Solve the differential equations

$$\begin{aligned} y_1' &= -2y_1 - 5y_2 \\ y_2' &= y_1 + 4y_2 \end{aligned}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

Answer: We start by finding the eigenvalues and eigenvectors of the matrix of coefficients:

$$\begin{vmatrix} -2 - \lambda & -5 \\ 1 & 4 - \lambda \end{vmatrix} = (-2 - \lambda)(4 - \lambda) + 5 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

Therefore, the eigenvalues are $\lambda = 3$ and $\lambda = -1$.

We compute the eigenvector corresponding to the eigenvalue $\lambda = 3$: $\begin{bmatrix} -5 & -5 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Therefore, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector. Corresponding to the eigenvalue $\lambda = -1$, we have $\begin{bmatrix} -1 & -5 \\ 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$, and so $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ is an eigenvector.

The general solution has the form $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t}$.

The initial conditions tell us to substitute $t = 0$, $y1(0) = 3$, and $y2(0) = 2$. That yields $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. We then have $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 5 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{13}{4} \\ -\frac{5}{4} \end{bmatrix}$. In the end, we get

$$\begin{aligned} y_1 &= -\frac{13}{4} e^{3t} + \frac{25}{4} e^{-t} \\ y_2 &= \ \ \frac{13}{4} e^{3t} - \frac{5}{4} e^{-t} \end{aligned}$$

10. Solve the differential equations

$$\begin{aligned} y_1' &= 3y_1 - 2y_2 \\ y_2' &= 2y_1 + 3y_2 \end{aligned}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

Answer: We again start by finding the eigenvalues and eigenvectors of the coefficient matrix:

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 4 = \lambda^2 - 6\lambda + 13.$$

We find that $\lambda = \frac{6 \pm \sqrt{-16}}{2} = 3 \pm 2i$. We compute an eigenvector corresponding to $3 + 2i$: $\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$, so an eigenvector is given by $\begin{bmatrix} i \\ 1 \end{bmatrix}$. Corresponding to the eigenvalue $3 - 2i$ we get the eigenvector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

We in fact only need one of these (so we'll use the first one), and plug into the formula on page 359, with $a = 3$, $b = 2$, and $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$. We see that the general solution has the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) \right) e^{3t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin(2t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) \right) e^{3t}$$

To find c_1 and c_2, we substitute $t = 0$, and get that $c_2 = 3$ and $c_1 = 2$. So the general solution is

$$\begin{aligned} y_1 &= e^{3t}(-2\sin(2t) + 3\cos(2t)) \\ y_2 &= e^{3t}(2\cos(2t) + 3\sin(2t)) \end{aligned}$$