MT216.03: Introduction to Abstract Mathematics
Examination 1
Answers

1. *(20 points)* Let d be the greatest common divisor of 24 and 37. Use the Euclidean algorithm to find d, and to find integers a and b so that $24a + 37b = d$.

Answer: We have

$$\begin{aligned} 37 &= 1\cdot 24 + 13 \\ 24 &= 1\cdot 13 + 11 \\ 13 &= 1\cdot 11 + \ \ 2 \\ 11 &= 5\cdot \ \ 2 + \ \ 1 \\ 2 &= 2\cdot \ \ 1 \end{aligned}$$

Therefore,

$$\begin{aligned} 1 &= 1\cdot 11 &&+ (-5)\cdot (2) \\ &= 1\cdot 11 &&+ (-5)\cdot (13 - 11) \\ &= 6\cdot 11 &&+ (-5)\cdot (13) \\ &= 6\cdot (24-13) &&+ (-5)\cdot (13) \\ &= 6\cdot 24 &&+ (-11)\cdot (13) \\ &= 6\cdot 24 &&+ (-11)\cdot (37-24) \\ &= 17\cdot 24 &&+ (-11)\cdot (37) \end{aligned}$$

2. *(20 points)* Define a sequence of real numbers with the definitions

$$\begin{aligned} x_1 &= 1 \\ x_n &= \sqrt{x_{n-1} + 1} \end{aligned}$$

Prove by induction that $x_n < 2$ for all positive integers n.

Answer: We can see that $x_1 < 2$. Now, assuming that $x_k < 2$, we have

$$\begin{aligned} x_k + 1 &< 3 \\ \sqrt{x_k + 1} &< \sqrt{3} < 2 \\ x_{k+1} &< 2 \end{aligned}$$

That concludes the induction.

3. *(20 points)* Let r and n be non-negative integers. Prove using induction that

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}.$$

Answer: We use induction on n. When $n = 0$, the left-hand side of the formula gives $\binom{r}{0} = 1$, and the right-hand side gives $\binom{r+1}{0} = 1$.

Now, assuming that

$$\sum_{k=0}^{s} \binom{r+k}{k} = \binom{r+s+1}{s},$$

we have

$$\sum_{k=0}^{s+1} \binom{r+k}{k} = \sum_{k=0}^{s} \binom{r+k}{k} + \binom{r+s+1}{s+1} = \binom{r+s+1}{s} + \binom{r+s+1}{s+1} = \binom{r+s+2}{s+1},$$

which establishes the induction.

4. *(20 points)* Find four different complex numbers z so that $z^4 = -2$. Express each value of z explicitly in terms of radicals.

Answer: We write $z = r(\cos\theta + i\sin\theta)$, so that $z^4 = r^4(\cos 4\theta + i\sin 4\theta)$. We therefore know that $r^4 = 2$, $\cos 4\theta = -1$, and $\sin 4\theta = 0$, telling us immediately that $r = \sqrt[4]{2}$.

One solution to the trigonometric equations is $4\theta = \pi$, meaning that $\theta = \frac{\pi}{4}$, and $z = \sqrt[4]{2}(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}) = \frac{\sqrt[4]{8}}{2} + \frac{i\sqrt[4]{8}}{2}$.

A second solution is $4\theta = 3\pi$, meaning that $\theta = \frac{3\pi}{4}$, and $z = \sqrt[4]{2}(-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}) = \frac{-\sqrt[4]{8}}{2} + \frac{i\sqrt[4]{8}}{2}$.

A third solution is $4\theta = 5\pi$. Then $\theta = \frac{5\pi}{4}$, and $z = \sqrt[4]{2}(-\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}) = \frac{-\sqrt[4]{8}}{2} - \frac{i\sqrt[4]{8}}{2}$.

The fourth solution is $4\theta = 7\pi$. Then $\theta = \frac{7\pi}{4}$, and $z = \sqrt[4]{2}(\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}) = \frac{\sqrt[4]{8}}{2} - \frac{i\sqrt[4]{8}}{2}$.

5. *(20 points)* Suppose that k and n are integers, with $n \geq 2$ and $k \geq 0$. Prove that

$$F_n F_{n+k} - F_{n-1} F_{n+k+1} = (-1)^{n+1} F_{k+1}.$$

Answer: We proceed by induction on n. When $n = 2$, the left-hand side of the formula is $F_2 F_{k+2} - F_1 F_{k+3} = F_{k+2} - F_{k+3} = -(F_{k+3} - F_{k+2})$, and the right-hand side gives $(-1)^3 F_{k+1} = -F_{k+1}$. Because $F_{k+3} - F_{k+2} = F_{k+1}$, this formula is correct.

Now, assume that

$$F_r F_{r+k} - F_{r-1} F_{r+k+1} = (-1)^{r+1} F_{k+1}.$$

We compute

$$\begin{aligned}
F_{r+1} F_{r+1+k} - F_r F_{r+k+2} &= F_{r+1} F_{r+1+k} - F_r (F_{r+k} + F_{r+k+1}) \\
&= F_{r+1} F_{r+1+k} - F_r F_{r+k} - F_r F_{r+k+1} \\
&= (F_{r+1} - F_r) F_{r+1+k} - F_r F_{r+k} \\
&= F_{r-1} F_{r+1+k} - F_r F_{r+k} \\
&= -(F_r F_{r+k} - F_{r-1} F_{r+1+k}) = -((-1)^{r+1} F_{k+1}) = (-1)^{r+2} F_{k+1}.
\end{aligned}$$

That concludes the induction.

Grade	Number of people
100	3
95	1
90	1
85	1
80	1
75	1
70	1
63	1
60	1
58	1
50	1
15	1

Mean: 74.36

Standard deviation: 23.19