MT216.03: Introduction to Abstract Mathematics
Examination 2
Answers

1. (*15 points*) Let n and k be integers, with $n \geq 0$.

(*a*) Give the complete definition of the binomial coefficient $\binom{n}{k}$.

(*b*) State the binomial theorem.

(*c*) What is the coefficient of x^{18} in $(x+2)^{20}$? State your answer both as a binomial coefficient and as a positive integer.

Answer: (*a*) If $k < 0$ or $k > n$, we have $\binom{n}{k} = 0$, and otherwise $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

(*b*) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

(*c*) The coefficient is $2^2 \binom{20}{18} = 4 \cdot \dfrac{20 \cdot 19}{2} = 4 \cdot 190 = 760$.

2. (*20 points*) Let $f : X \to Y$ be a function. Suppose that for all subsets $A, B \subset X$, we know that $f(A \cup B) = f(A) \cup f(B)$. Prove or give a counterexample:

(*a*) f must be a surjection.

(*b*) f must be an injection.

Answer: We proved in class that $f(A \cup B) = f(A) \cup f(B)$ for all functions, so both statements are *false*.

(*a*) A counterexample is given by $f : \{1\} \to \{1, 2\}$ with $f(1) = 1$. This function is not a surjection, and because $\{1\}$ has only two subsets—$\{1\}$ and $\emptyset$—it is easy to verify that $f(A \cup B) = f(A) \cup f(B)$ for all subsets.

(*b*) A counterexample is given by $f : \{1, 2\} \to \{1\}$ with $f(1) = f(2) = 1$. Because the only value taken by the function is 1, it is clear that $f(A) \cup f(B) = f(A \cup B)$.

3. (*20 points*) Use the Chinese Remainder Theorem to find the smallest positive integer n so that

- n has a remainder of 15 when divided by 19.
- n has a remainder of 12 when divided by 21.
- n is odd.

Answer: We need to solve

$$\begin{aligned} n &\equiv 15 \pmod{19} \\ n &\equiv 12 \pmod{21} \\ n &\equiv \ \,1 \pmod{\ 2} \end{aligned}$$

The first tells us that $n = 19a + 15$, and substitution into the second congruence gives $19a + 15 \equiv 12 \pmod{21}$, or $19a \equiv -3 \pmod{21}$. This is the same as $-19a \equiv 3 \pmod{21}$, or $2a \equiv 3 \pmod{21}$, or $2a \equiv 24 \pmod{21}$, meaning that $a \equiv 12 \pmod{21}$. We now know that $a = 21b + 12$, and so $n = 19(21b + 12) + 15 = 19 \cdot 21b + 19 \cdot 12 + 15 = 19 \cdot 21b + 228 + 15 = 19 \cdot 21b + 243$. Now, $19 \cdot 21b + 243 \equiv 1 \pmod{2}$ says that $b \equiv 0 \pmod{2}$. Set $b = 2c$, and we have $n = 2 \cdot 19 \cdot 21c + 243$, and the smallest positive integer satisfying all three congruences is 243.

4. (*30 points*) Suppose that A and B are sets, and $f : A \to B$ any function. Define a relation on A by saying that if $a_1, a_2 \in A$, then $a_1 \sim a_2$ if $f(a_1) = f(a_2)$.

   (a) Show that this relation is an equivalence relation.

   (b) Take the specific example of $A = \mathbf{C}$, $B = \mathbf{C}$, and $f(z) = z^4$. What is the equivalence class of the number $1 + i$ under this equivalence relation?

Answer: (a) An equivalence relation is reflexive, symmetric, and transitive. The first says that $a \sim a$ for all $a \in A$. The second says that if $a \sim b$, then $b \sim a$. The third says that if $a \sim b$ and $b \sim c$, then $a \sim c$.

(b) Because $f(a) = f(a)$, we have $a \sim a$. If $a \sim b$, then $f(a) = f(b)$, so $f(b) = f(a)$, and then $b \sim a$. If $a \sim b$ and $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$, and then $a \sim c$.

(c) If $z \sim 1 + i$, then $z^4 = (1 + i)^4 = (2i)^2 = -4$. The four elements of the equivalence class are $1 + i$, $1 - i$, $-1 + i$, and $-1 - i$.

5. (*5 points*) Suppose that $h : Y \to Z$ is a function, and $B \subseteq Z$. What is the definition of the set $h^{-1}(B)$?

Answer: We have $h^{-1}(B) = \{y \in Y : h(y) \in B\}$.

6. (*10 points*) Suppose that ζ is a root of unity.

   (a) Define what is meant by the *order* of ζ.

   (b) Suppose that ζ has order 60. What is the order of ζ^{36}?

Answer: (a) The order of ζ is the smallest positive integer k so that $\zeta^k = 1$.

(b) We know that if $o(\zeta) = n$, then $o(\zeta^a) = n/(n, a)$. In this case, the formula gives $o(\zeta^{36}) = 60/(36, 60) = 5$.

Grade	Number of people
84	1
83	1
79	1
75	1
73	2
67	1
66	2
65	1
63	1
50	1
44	1
10	1

Mean: 64.14
Standard deviation: 18.52