Name____________________________________

1	2	3	4	5	6	7	8	9	10	11	12	Total

MT216.03: Introduction to Abstract Mathematics
Final Examination
Friday, May 11, 2012, 12:30 PM
Carney 204A

Please label your answers clearly, as I will not have time to perform extensive searches for answers. No credit will be given for answers without explanations.

Cheating will result in a failing grade.

Calculators may not be used during this examination.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. (*10 points*) Use the Euclidean algorithm to find the smallest positive integer n so that

$$31n \equiv 4 \pmod{43}$$

or prove that there are no solutions.

2. (*5 points*) Find a non-constant monic sixth degree polynomial $f(x) \in \mathbf{Z}[x]$ so that:
   - $f(3) = f'(3) = 0$
   - $f(5) = f'(5) = f''(5) = 0$
   - $f(6) = 0$

or prove that no such polynomial exists. You can state your answer as a product of irreducible polynomials.

3. (*5 points*) On my calculator, I can compute that

$$\begin{aligned} 2^{8910} &\equiv 1 \pmod{8911} \\ 3^{8910} &\equiv 1 \pmod{8911} \\ (2, 8911) &= 1 \\ (3, 8911) &= 1 \end{aligned}$$

Based on these computations, which of the following conclusions can be drawn?

(a) 8911 is definitely composite.
(b) 8911 is definitely prime.
(c) 8911 could be either prime or composite.

Be sure to explain your answer.

4. (*10 points*) A horde of 12 Mongol invaders has raided a castle, and found a treasure trove of gold coins. The invaders attempt to divide the pile of coins evenly, and find that there are 3 remaining. A mêlée ensues, and 5 of the invaders die. The remaining 7 members of the horde try again to divide the coins evenly, and now there are 4 coins left over. Another fight ensues, and 2 more invaders die. The remaining 5 marauders now divide the pile evenly.

What is the smallest number of coins that could be in the trove?

5. (*10 points*) Let n be a positive integer. Prove using induction that

$$\int_0^1 (1-x^2)^n \, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Hint: Use the identity $(1-x^2)^n = (1-x^2)^{n-1} - x^2(1-x^2)^{n-1}$ and integrate by parts.

6. (*10 points*) Remember that the set μ_{1000} is defined by $\mu_{1000} = \{z \in \mathbf{C} \mid z^{1000} = 1\}$. Let j be a positive integer, and define $f : \mu_{1000} \to \mu_{1000}$ with the formula $f(x) = x^j$.

(*a*) If $(j, 1000) = 1$, prove that f is a bijection.

(*b*) If $(j, 1000) \neq 1$, prove that f is not a bijection.

7. (*10 points*) Define a sequence $\{x_n\}$ with the formulas

$$x_1 = 2$$
$$x_{n+1} = \sqrt{4 + x_n} \qquad n \geqslant 1$$

For example, $x_2 = \sqrt{6}$ and $x_3 = \sqrt{4 + \sqrt{6}}$.

Prove

(*a*) $x_n \leqslant 3$.

(*b*) $x_n \leqslant x_{n+1}$.

8. (*5 points*) Find 3 complex numbers which solve the equation $z^3 = 7$. Write each of those numbers in the form $a + bi$, where a and b are real numbers expressed using radicals.

9. (*5 points*) There are 9 monic quadratic polynomials in $\mathbf{F}_3[x]$. List all 9 of these polynomials, and indicate which are irreducible.

10. (*10 points*) As usual, define the Fibonacci numbers by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$. Let $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$. You may use the facts that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$.

Prove using induction that

$$F_{n+1} = \alpha F_n + \beta^n$$

for $n \geqslant 1$.

11. (*10 points*) Suppose that n and k are positive integers, with $n > k > 1$. Prove using the definition of binomial coefficient that

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}.$$

12. (*10 points*) Suppose that F is a field, and $f(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 \in F[x]$, with $n \geqslant 2$. Suppose that $\gamma_1, \gamma_2, \ldots, \gamma_n \in F$ are the n roots of $f(x)$. Prove that

$$c_0 = (-1)^n \gamma_1 \gamma_2 \cdots \gamma_n$$

and

$$c_{n-1} = -(\gamma_1 + \gamma_2 + \cdots + \gamma_n)$$