Mathematics 216
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Homework 1
Answers

1. Prove that $\sqrt{10}$ is irrational by using an even–odd argument.

Answer: Suppose that $\sqrt{10} = \frac{a}{b}$, where $\frac{a}{b}$ is in lowest terms. Squaring and cross-multiplying gives $10b^2 = a^2$. Because $10b^2$ is even, we know that a^2 is even. If a were odd, then a^2 would be odd, so we know that a is even.

Let $a = 2c$, and then substitution yields $10b^2 = 4c^2$, or $5b^2 = 2c^2$. Now we see that $2c^2$ is even, so $5b^2$ is even. If b were odd, then $5b^2$ would be odd, and therefore we know that b is even.

But if a and b are both even, then $\frac{a}{b}$ is not in lowest terms. This is a contradiction.

2. Prove that $\sqrt{7}$ is irrational.

Answer: Suppose that $\sqrt{7} = \frac{a}{b}$, with $\frac{a}{b}$ in lowest terms. Squaring and cross-multiplying yields $7b^2 = a^2$. If a were even, we could conclude that b is even, and *vice versa*. Therefore, both a and b are odd.

Let $a = 2c + 1$ and $b = 2d + 1$, so we have $7(2d + 1)^2 = (2c + 1)^2$. This becomes $28d^2 + 28d + 7 = 4c^2 + 4c + 1$, or $6 = 4c^2 + 4x - 28d^2 - 28d$. Division by 2 now yields $3 = 2c^2 + 2c - 14d^2 - 14d = 2(c^2 + c - 7d^2 - 7d)$. Now, the number 3 is odd, while the right-hand side of the equation is even, which is a contradiction. Therefore, $\sqrt{7}$ is irrational.

3. Prove that $\sqrt{90}$ is irrational without using the words "even" or "odd."

Answer: Suppose that $\sqrt{90} = \frac{a}{b}$. This is the same as $3\sqrt{10} = \frac{a}{b}$, or $\sqrt{10} = \frac{a}{3b}$. But we proved above that $\sqrt{10}$ is irrational, so we have arrived at a contradiction.

4. Prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Answer: We proceed by induction. When $n = 1$, we have $1^2 = \frac{(1)(2)(3)}{6}$, which is true.

Now, we assume that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}, \tag{1}$$

and we must prove that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Take (1) and add $(k+1)^2$ to both sides of the equation. We get

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\frac{k(2k+1) + 6(k+1)}{6} \\ &= (k+1)\frac{2k^2 + 7k + 6}{6} = (k+1)\frac{(2k+3)(k+2)}{6} \end{aligned}$$

which is the desired result.