Mathematics 216
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Homework 2
Answers

1. Prove using induction that
$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$$
for $n \geq 0$.

Answer: When $n = 0$, the left-hand side of the formula is $\sum_{k=0}^{0} x^k = x^0 = 1$, and the right-hand side is $\frac{x^1-1}{x-1} = 1$, so the formula checks.

Now, suppose that we know that
$$\sum_{k=0}^{r} x^k = \frac{x^{r+1}-1}{x-1},$$
and we want to prove that
$$\sum_{k=0}^{r+1} x^k = \frac{x^{r+2}-1}{x-1}.$$
We have
$$\sum_{k=0}^{r+1} x^k = \sum_{k=0}^{r} x^k + x^{r+1} = \frac{x^{r+1}-1}{x-1} + x^{r+1} = \frac{x^{r+1}-1}{x-1} + \frac{x^{r+2}-x^{r+1}}{x-1}$$
$$= \frac{x^{r+2}-1}{x-1},$$
which is the desired result.

2. Find a value of N so that $n^3 < 3^n$ if $n \geq N$, and prove that the inequality is true by using induction.

Answer: Trial and error shows that the formula seems to be true if $n \geq 4$, so we take $N = 4$. Proceeding by induction, we verify that the formula is indeed true when $n = 4$. Next, we assume that $k^3 < 3^k$, and we need to prove that $(k+1)^3 < 3^{k+1}$, where $k \geq 4$. Note that $1 + \frac{1}{k} \leq \frac{5}{4}$, and so $\left(1+\frac{1}{k}\right)^3 \leq (\frac{5}{4})^3 = \frac{125}{64} < 3$.

We take $k^3 < 3^k$, and multiply by $(1+\frac{1}{k})^3 < 3$, and we end with $(k+1)^3 < 3 \cdot 3^k = 3^{k+1}$.

3. Prove that $F_n < \left(\frac{5}{3}\right)^n$.

Answer: We proceed by induction. When $n = 1$, we have $F_1 = 1 < \frac{5}{3}$, which is true. When $n = 2$, we have $F_2 = 1 < (\frac{5}{3})^2$, which is also true.

Now, we suppose that $F_k < (\frac{5}{3})^k$ and $F_{k-1} < (\frac{5}{3})^{k-1}$, and we add and use the fact that $\frac{8}{5} < \frac{5}{3}$:
$$F_{k+1} = F_k + F_{k-1} < \left(\frac{5}{3}\right)^k + \left(\frac{5}{3}\right)^{k-1} = \left(\frac{5}{3}\right)^k \left(1 + \frac{3}{5}\right) = \left(\frac{5}{3}\right)^k \left(\frac{8}{5}\right) < \left(\frac{5}{3}\right)^{k+1}.$$