Mathematics 216
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Homework 3
Answers

1. Find a value of N so that $F_n > \left(\frac{3}{2}\right)^n$ if $n > N$, and then prove that the inequality is true by using induction.

Answer: Finding the value of N requires a lot of trial and error, or else the use of a computer. I compute that $F_{10} = 55$, $(\frac{3}{2})^{10} = \frac{59049}{1024} \approx 57.7$, $F_{11} = 89$, $(\frac{3}{2})^{11} = \frac{177147}{2048} \approx 86.5$, $F_{12} = 144$, and $(\frac{3}{2})^{12} = \frac{531441}{4096} \approx 129.7$.

We can be sure that $F_{11} > (\frac{3}{2})^{11}$ and $F_{12} > (\frac{3}{2})^{12}$. The induction proof continues by assuming that $F_k > (\frac{3}{2})^k$ and $F_{k+1} > (\frac{3}{2})^{k+1}$, and then adding these two inequalities yields $F_{k+2} > (\frac{3}{2})^k + (\frac{3}{2})^{k+1} = (\frac{3}{2})^{k+1}(\frac{2}{3}+1) = (\frac{3}{2})^{k+1}(\frac{5}{3}) > (\frac{3}{2})^{k+1}(\frac{3}{2}) = (\frac{3}{2})^{k+2}$. That concludes the induction argument.

2. Let n be a positive integer. Prove using induction (and l'Hôpital's rule) that

$$\lim_{x\to\infty} \frac{(\log x)^n}{x} = 0.$$

Answer: We proceed by induction. To handle the case $n = 1$, we use l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{x^{-1}}{1} = \lim_{x\to\infty} x^{-1} = 0.$$

Now, we assume that

$$\lim_{x\to\infty} \frac{(\log x)^k}{x} = 0,$$

and apply l'Hôpital's rule to

$$\lim_{x\to\infty} \frac{(\log x)^{k+1}}{x} = \lim_{x\to\infty} \frac{(k+1)(\log x)^k x^{-1}}{1} = \lim_{x\to\infty} \frac{(k+1)(\log x)^k}{x} = \lim_{x\to\infty} (k+1)\frac{(\log x)^k}{x} = 0.$$

That concludes the induction.

3. Let n be a positive integer. Prove using induction that $n^3 + 2n$ is always a multiple of 3.

Answer: The case $n = 1$ is clear. Assuming that $k^3 + 2k$ is a multiple of 3, we compute $(k+1)^3 + 2(k+1) = (k^3 + 3k^2 + 3k + 1) + (2k + 2) = (k^3 + 2k) + 3(k^2 + k + 1)$. This expression is a sum of two multiples of 3, and hence is a multiple of 3.