Mathematics 216
Robert Gross
Homework 4
Answers

1. Let n be a positive integer. Prove using induction that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Answer: We first check that the statement is true for $n = 1$, using l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Next, we assume that

$$\lim_{x\to\infty} \frac{x^k}{e^x} = 0$$

and compute

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} \frac{(k+1)x^k}{e^x} = (k+1) \lim_{x\to\infty} \frac{x^k}{e^x} = (k+1)(0) = 0.$$

That completes the induction.

2. The Gamma function is defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x \geq 1$. This is an improper integral, and for our purposes, you may assume that the integral converges with $x \geq 1$. Prove that $\Gamma(1) = 1$.

Answer: We have

$$\Gamma(1) = \int_0^\infty e^{-t} dt = \lim_{t\to\infty} -e^{-t}\Big]_0^t = 1 - \lim_{t\to\infty} \frac{1}{e^t} = 1 - 0 = 0.$$

3. Use integration by parts, along with problem 1, to prove that $\Gamma(n+1) = n\Gamma(n)$ if n is a positive integer.

Answer: For integration by parts, we set $u = t^n$, $dv = e^{-t}\, dt$, $du = nt^{n-1}$, and $v = -e^{-t}$. We have

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt = -\frac{t^n}{e^t}\Big]_0^\infty + \int_0^\infty nt^{n-1} e^{-t}\, dt = 0 + n\Gamma(n-1).$$

Here we compute

$$-\frac{t^n}{e^t}\Big]_0^\infty = -\lim_{t\to\infty} \frac{t^n}{e^t} + \frac{0^n}{e^0} = 0 + 0 = 0$$

by using problem 1.

4. Prove using induction that if n is a positive integer, then $\Gamma(n) = (n-1)!$.

Answer: We know that $\Gamma(1) = 1$ and $0! = 1$, so the equation is true when $n = 1$.

Now, assuming that $\Gamma(k) = (k-1)!$, we compute $\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k!$, which completes the induction.

5. Let n be a positive integer. Show that

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1.$$

Answer: We proceed by induction. When $n = 1$, the left-hand side of the equation is F_1, which is 1, and the right-hand side is $F_3 - 1 = 2 - 1 = 1$. The equation is true when $n = 1$.

Now, assume that

$$\sum_{k=1}^{r} F_k = F_{r+2} - 1.$$

We compute

$$\sum_{k=1}^{r+1} F_k = \sum_{k=1}^{r} F_k + F_{r+1} = (F_{r+2} - 1) + F_{r+1} = F_{r+3} - 1.$$

That completes the induction.