Mathematics 216
Robert Gross
Homework 5
Answers

1. Let n be a positive integer. Prove using induction that

$$\lim_{x\to 0^+} x(\log x)^n = 0.$$

The notation $\lim_{x\to 0^+}$ means that x tends to 0 and is positive. The inequality $x > 0$ is required because $\log x$ is only defined for positive x. *Hint*: Apply l'Hôpital's rule, but make sure that you do it correctly.

Answer: First, we verify that the equation holds for $n = 1$:

$$\lim_{x\to 0^+} x(\log x) = \lim_{x\to 0^+} \frac{\log x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} -x = 0.$$

Next, we assume that $\lim_{x\to 0^+} x(\log x)^k = 0$, and we evaluate

$$\lim_{x\to 0^+} x(\log x)^{k+1} = \lim_{x\to 0^+} \frac{(\log x)^{k+1}}{x^{-1}} = \lim_{x\to 0^+} \frac{(k+1)(\log x)^k x^{-1}}{-x^{-2}} = -(k+1) \lim_{x\to 0^+} x(\log x)^k = 0.$$

2. Use an even–odd argument to show that $\sqrt{13}$ is irrational. *Hint*: This is a bit tricky, and requires a bit more thought than our previous irrationality proofs.

Answer: Suppose that $\sqrt{13} = \frac{a}{b}$, with $\frac{a}{b}$ in lowest terms. That means that $13b^2 = a^2$, and the usual arguments show that if a were even, then b would be even, and *vice versa*. Therefore, we know that a and b are both odd.

Set $a = 2c + 1$ and $b = 2d + 1$, and we have $13(4d^2 + 4d + 1) = 4c^2 + 4c + 1$. This becomes $52d^2 + 52d + 13 = 4c^2 + 4c + 1$. Subtract 1 from both sides of the equation, and divide by 4, and we have $13(d^2 + d) + 1 = c^2 + c$.

Finding the contradiction in this equation is subtle. The point is that $d^2 + d = d(d + 1)$, and therefore $d^2 + d$ is the product of two consecutive integers and is always even. Hence, $13(d^2 + d)$ is also always even, so $13(d^2 + d) + 1$ is always odd. However, $c^2 + c = c(c + 1)$ is also the product of two consecutive integers, and therefore $c^2 + c$ is always even. This is the contradiction: the number on the left-hand side of the equals sign is odd, and the number on the right-hand side is even.

3. Find the smallest positive integer N so that $n^3 \le 2^n$ if $n \ge N$, and prove your result using induction.

Answer: A bit of experimentation shows that $9^3 > 2^9$, but $10^3 < 2^{10}$, so we can take $N = 10$. This computation also checks the starting step of the induction.

Now, assuming that $k^3 \le 2^k$, we must prove that $(k + 1)^3 \le 2^{k+1}$. One method is to show that $(k + 1)^3/k^3 \le 2$, and then multiply by $k^3 < 2^k$ to get the desired result.

Here's another approach. We know that $k \ge 10$, and therefore

- $3k + 1 \le 4k$.
- $4k < 4k^2$.
- $3k^2 + 3k + 1 < 3k^2 + 4k^2 = 7k^2 < k^3$.
- $k^3 + 3k^2 + 3k + 1 < k^3 + k^3 = 2k^3$.

Hence $(k+1)^3 < 2k^3 < 2(2^k) = 2^{k+1}$.

4. Let $n > 2$ be an integer. Show that

$$F_n F_{n+1} - F_{n-1} F_{n+2} = (-1)^n.$$

Answer: We start by checking the case $n = 3$. (The equality is actually true when $n = 2$ as well, but the problem accidentally specified $n > 2$.) We have $F_3F_4 - F_2F_5 = 2\cdot 3 - 1\cdot 5 = 1 = (-1)^{3+1}$, so the equation is true when $n = 3$.

Now, we assume that

$$F_k F_{k+1} - F_{k-1} F_{k+2} = (-1)^{k+1},$$

and we must prove that

$$F_{k+1} F_{k+2} - F_k F_{k+3} = (-1)^{k+2}.$$

We have

$$\begin{aligned} F_{k+1}F_{k+2} - F_kF_{k+3} &= F_{k+1}F_{k+2} - F_k(F_{k+2} + F_{k+1}) = F_{k+2}(F_{k+1} - F_k) - F_kF_{k+1} \\ &= F_{k+2}F_{k-1} - F_kF_{k+1} = -(F_kF_{k+1} - F_{k+2}F_{k-1}) = -(-1)^{k+1} = (-1)^{k+2}. \end{aligned}$$

That concludes the induction.