Mathematics 216
Robert Gross
Homework 6
Answer

1. Let n be a positive integer. Prove using induction and integration by parts that

$$\int_0^1 (-\log x)^n \, dx = n!.$$

This is an improper integral, so you will need to explain how you evaluated the lower limit when applying integration by parts.

Answer: First, we deal with the case $n = 1$. We have

$$\int_0^1 -\log x \, dx = x - x\log x\Big|_0^1 = 1 - 1\log 1 - (0 - \lim_{t\to 0^+} t\log t) = 1 = 1!.$$

Here we used the equation $\lim_{t\to 0^+} t\log t = 0$ computed in the last homework assignment.

Second, we assume that

$$\int_0^1 (-\log x)^k \, dx = k!,$$

and we compute

$$\int_0^1 (-\log x)^{k+1} \, dx = x(-\log x)^{k+1}\Big|_0^1 + (k+1)\int_0^1 (-\log x)^k \, dx.$$

Here we have used integration by parts, with $u = (-\log x)^{k+1}$, $du = -(k+1)(-\log x)^k x^{-1}\, dx$, $dv = dx$, and $v = x$.

Now,

$$x(-\log x)^{k+1}\Big|_0^1 = 1(-\log 1)^{k+1} - \lim_{t\to 0^+} t(-\log t)^{k+1} = 0 - 0 = 0,$$

relying on the limit $\lim_{t\to 0^+} t(\log t)^n = 0$ from the previous homework.

We now have

$$\int_0^1 (-\log x)^{k+1} \, dx = (k+1)\int_0^1 (-\log x)^k \, dx = (k+1)k! = (k+1)!,$$

completing the induction.

2. Let m and n be positive integers, with $m \le n$. Prove that

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}.$$

Answer: We proceed by induction on n. The case $n = m$ produces the equation $\binom{m}{m} = \binom{m+1}{m+1}$, which is true because both sides are equal to 1.

Second, we assume that

$$\sum_{k=m}^{p} \binom{k}{m} = \binom{p+1}{m+1},$$

and we compute

$$\sum_{k=m}^{p+1} \binom{k}{m} = \sum_{k=m}^{p} \binom{k}{m} + \binom{p+1}{m} = \binom{p+1}{m+1} + \binom{p+1}{m} = \binom{p+2}{m+1},$$

which establishes the induction.

3. Let a and b be positive numbers, with $b > 1$. Prove that

$$F_a F_{b-1} + F_{a+1} F_b = F_{a+b}.$$

Answer: We proceed by induction on a. When $a = 1$, we have to verify the equation

$$F_1 F_{b-1} + F_2 F_b = F_{b+1}.$$

Because $F_1 = F_2 = 1$, this is just the equation $F_{b-1} + F_b = F_{b+1}$, which is part of the definition of the Fibonacci numbers.

We will need 2 assumptions for the proof to proceed, so we also need to verify the assertion when $a = 1$. That equation is

$$F_2 F_{b-1} + F_3 F_b = F_{b+2}.$$

We expand the left-hand side, and get $F_{b-1} + 2F_b = (F_{b-1} + F_b) + F_b = F_{b+1} + F_b = F_{b+2}$, thereby verifying the equation when $a = 2$.

The actual induction is now quite easy. We assume that the statement is true when $a = k - 1$ and when $a = k$, and add the two assumptions together:

$$\begin{array}{rcrl} F_{k-1}F_{b-1} & + & F_k F_b & = F_{k-1+b} \\ F_k F_{b-1} & + & F_{k+1} F_b & = F_{k+b} \end{array}$$

Now, adding these two equations yields on the left-hand side $(F_{k-1}F_{b-1} + F_k F_{b-1}) + (F_k F_b + F_{k+1} F_b) = (F_{k-1} + F_k)F_{b-1} + (F_k + F_{k+1})F_b = F_{k+1}F_{b-1} + F_{k+2}F_b$, and on the right-hand side we get $F_{k+b-1} + F_{k+b} = F_{k+b+1}$. This means that we have proven $F_{k+1}F_{b-1} + F_{k+2}F_b = F_{k+b+1}$, and that in turn is the desired result when $a = k + 1$.