1. If $f(x)$ is a function of $x$, write $f^{(n)}$ to refer to the $n$th derivative of $f$ with respect to $x$. If, for example, $f(x) = \sin x$, then $f^{(5)}(x) = \cos x$ and $f^{(6)}(x) = -\sin x$. We define $f^{(0)}(x)$ to be $f(x)$.

Suppose that $u(x)$ and $v(x)$ are functions of $x$. To save space, we will just write $u$ and $v$ rather than $u(x)$ and $v(x)$. Suppose that $n$ is a positive integer. Prove that

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}.$$ 

In other words, you will prove that

$$\frac{d^n}{dx^n} (uv) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{d^k u}{dx^k} \right) \left( \frac{d^{n-k} v}{dx^{n-k}} \right).$$

*Hint:* Proceed as in the proof of the binomial theorem.

*Answer:* We proceed by induction. When $n = 1$, the left-hand side of the formula is $(uv)'$, and the right-hand side is $uv' + u'v$. The product rule tells us that $(uv)' = uv' + u'v$, so the formula is true when $n = 1$.

Now, we assume that

$$(uv)^{(p)} = \sum_{k=0}^{p} \binom{p}{k} u^{(k)} v^{(p-k)}.$$ 

Differentiating both sides of the equation and then re-arranging yields:

$$(uv)^{(p+1)} = \sum_{k=0}^{p} \frac{d}{dx} \left( \binom{p}{k} u^{(k)} v^{(p-k)} \right) = \sum_{k=0}^{p} \binom{p}{k} \left( u^{(k+1)} v^{(p-k)} + u^{(k)} v^{(p-k+1)} \right)$$

$$= \sum_{k=0}^{p} \binom{p}{k} u^{(k+1)} v^{(p-k)} + \sum_{k=0}^{p} \binom{p}{k} u^{(k)} v^{(p-k+1)}$$

$$= u^{(p+1)} v^{(0)} + \sum_{k=0}^{p-1} \binom{p}{k} u^{(k+1)} v^{(p-k)} + \sum_{k=1}^{p} \binom{p}{k} u^{(k)} v^{(p-k+1)} + u^{(0)} v^{(p+1)}$$

[Re-index the first sum, using $k = t - 1; t = k + 1$:

$$= u^{(p+1)} v^{(0)} + \sum_{t=1}^{p} \binom{p}{t-1} u^{(t)} v^{(p+1-t)} + \sum_{k=1}^{p} \binom{p}{k} u^{(k)} v^{(p+1-k)} + u^{(0)} v^{(p+1)}$$

$$= u^{(p+1)} v^{(0)} + \sum_{r=1}^{p} \left( \binom{p}{r-1} + \binom{p}{r} \right) u^{(t)} v^{(p+1-t)} + u^{(0)} v^{(p+1)}$$

$$= u^{(p+1)} v^{(0)} + \sum_{r=1}^{p} \binom{r+1}{r} u^{(t)} v^{(p+1-t)} + u^{(0)} v^{(p+1)}$$

$$= \sum_{r=0}^{p+1} \binom{p+1}{r} u^{(t)} v^{(p+1-t)}.$$
2. Use the previous formula to compute $\frac{d^7}{dx^7} \left( x^2 e^{3x} \right)$.

*Answer:* We write $u = x^2$, $u' = 2x$, $u'' = 2$, and $u^{(k)} = 0$ if $k \geq 3$. We also have $v = e^{3x}$, $v' = 3e^{3x}$, $v'' = 3^2 e^{3x}$, and in general $v^{(k)} = 3^k e^{3x}$. The previous problem now tells us that

$$\frac{d^7}{dx^7} \left( x^2 e^{3x} \right) = \left( \begin{array}{c} 7 \\ 0 \end{array} \right) u v^{(7)} + \left( \begin{array}{c} 7 \\ 6 \end{array} \right) u' v^{(6)} + \left( \begin{array}{c} 7 \\ 12 \end{array} \right) u'' v^{(5)}$$

$$= x^2 \cdot 3^7 e^{3x} + 7(2x) \cdot 3^6 e^{3x} + 21(2) \cdot 3^5 e^{3x}$$

$$= (3^7 x^2 + 14 \cdot 3^6 x + 14 \cdot 3^5) e^{3x}$$

3. Let $n$ be a positive integer. Show that

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

*Answer:* We check first that the equation is true when $n = 1$. The sum gives $F_1^2 = 1$, and the right-hand side of the equation is $F_1 F_2 = 1 \cdot 1 = 1$.

Now, we assume that the result is true when $n = r$, and we compute

$$\sum_{k=1}^{r+1} F_k^2 = \sum_{k=1}^{r} F_k^2 + F_{r+1}^2 = F_r F_{r+1} + F_{r+1}^2 = F_{r+1}(F_r + F_{r+1}) = F_{r+1} F_{r+2}.$$

That concludes the induction.