Mathematics 216
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Homework 7
Answers

1. If $f(x)$ is a function of x, write $f^{(n)}$ to refer to the nth derivative of f with respect to x. If, for example, $f(x) = \sin x$, then $f^{(5)}(x) = \cos x$ and $f^{(6)}(x) = -\sin x$. We define $f^{(0)}(x)$ to be $f(x)$.

Suppose that $u(x)$ and $v(x)$ are functions of x. To save space, we will just write u and v rather than $u(x)$ and $v(x)$. Suppose that n is a positive integer. Prove that

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}.$$

In other words, you will prove that

$$\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d^k u}{dx^k}\right) \left(\frac{d^{n-k} v}{dx^{n-k}}\right).$$

Hint: Proceed as in the proof of the binomial theorem.

Answer: We proceed by induction. When $n = 1$, the left-hand side of the formula is $(uv)'$, and the right-hand side is $uv' + u'v$. The product rule tells us that $(uv)' = uv' + u'v$, so the the formula is true when $n = 1$.

Now, we assume that

$$(uv)^{(p)} = \sum_{k=0}^{p} \binom{p}{k} u^{(k)} v^{(p-k)}.$$

Differentiating both sides of the equation and then re-arranging yields:

$$\begin{aligned}
(uv)^{p+1} &= \sum_{k=0}^{p} \frac{d}{dx}\left(\binom{p}{k} u^{(k)} v^{(p-k)}\right) = \sum_{k=0}^{p} \binom{p}{k} \left(u^{(k+1)} v^{(p-k)} + u^{(k)} v^{(p-k+1)}\right) \\
&= \sum_{k=0}^{p} \binom{p}{k} u^{(k+1)} v^{(p-k)} + \sum_{k=0}^{p} \binom{p}{k} u^{(k)} v^{(p-k+1)} \\
&= u^{(p+1)} v^{(0)} + \sum_{k=0}^{p-1} \binom{p}{k} u^{(k+1)} v^{(p-k)} + \sum_{k=1}^{p} \binom{p}{k} u^{(k)} v^{(p-k+1)} + u^{(0)} v^{(p+1)}
\end{aligned}$$

[Re-index the first sum, using $k = t - 1$; $t = k + 1$:]

$$\begin{aligned}
&= u^{(p+1)} v^{(0)} + \sum_{t=1}^{p} \binom{p}{t-1} u^{(t)} v^{(p+1-t)} + \sum_{k=1}^{p} \binom{p}{k} u^{(k)} v^{(p+1-k)} + u^{(0)} v^{(p+1)} \\
&= u^{(p+1)} v^{(0)} + \sum_{r=1}^{p} \left(\binom{p}{r-1} + \binom{p}{r}\right) u^{(t)} v^{(p+1-t)} + u^{(0)} v^{(p+1)} \\
&= u^{(p+1)} v^{(0)} + \sum_{r=1}^{p} \binom{p+1}{r} u^{(t)} v^{(p+1-t)} + u^{(0)} v^{(p+1)} \\
&= \sum_{r=0}^{p+1} \binom{p+1}{r} u^{(t)} v^{(p+1-t)}.
\end{aligned}$$

2. Use the previous formula to compute $\frac{d^7}{dx^7}\left(x^2e^{3x}\right)$.

Answer: We write $u = x^2$, $u' = 2x$, $u'' = 2$, and $u^{(k)} = 0$ if $k \geq 3$. We also have $v = e^{3x}$, $v' = 3e^{3x}$, $v'' = 3^2e^{3x}$, and in general $v^{(k)} = 3^ke^{3x}$. The previous problem now tells us that

$$\begin{aligned}\frac{d^7}{dx^7}\left(x^2e^{3x}\right) &= \binom{7}{0}uv^{(7)} + \binom{7}{1}u'v^{(6)} + \binom{7}{2}u''v^{(5)} \\ &= x^2 \cdot 3^7e^{3x} + 7(2x) \cdot 3^6e^{3x} + 21(2) \cdot 3^5e^{3x} \\ &= (3^7x^2 + 14 \cdot 3^6x + 14 \cdot 3^6)e^{3x}\end{aligned}$$

3. Let n be a positive integer. Show that

$$\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}.$$

Answer: We check first that the equation is true when $n = 1$. The sum gives $F_1^2 = 1$, and the right-hand side of the equation is $F_1F_2 = 1 \cdot 1 = 1$.

Now, we assume that the result is true when $n = r$, and we compute

$$\sum_{k=1}^{r+1} F_k^2 = \sum_{k=1}^{r} F_k^2 + F_{r+1}^2 = F_rF_{r+1} + F_{r+1}^2 = F_{r+1}(F_r + F_{r+1}) = F_{r+1}F_{r+2}.$$

That concludes the induction.