Mathematics 216
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Homework 8
Answers

1. Let n be an integer which is at least 2. Use integration by parts to derive the formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

Answer: To integrate by parts, we set $u = \cos^{n-1} x$, $du = -(n-1)\cos^{n-2} x \sin x\,dx$, $dv = \cos x\,dx$, and $v = \sin x$. We get

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \\
n\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx \\
\int \cos^n x\,dx &= \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.
\end{aligned}$$

2. Let n be an integer which is at least 1. Use induction and the previous problem to prove that

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Answer: We start by verifying the formula when $n = 1$. We compute, using the previous problem, that

$$\int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \left.\frac{\cos^2 x \sin x}{3}\right|_0^{\frac{\pi}{2}} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos x\,dx = 0 - 0 + \frac{2}{3}\sin x\Big|_0^{\frac{\pi}{2}} = \frac{2}{3}.$$

The left-hand side of the formula gives $\dfrac{2^2 1!^2}{3!} = \dfrac{4}{6} = \dfrac{2}{3}$.

Now, assuming that

$$\int_0^{\frac{\pi}{2}} \cos^{2p+1} x\,dx = \frac{2^{2p}(p!)^2}{(2p+1)!},$$

we compute

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^{2p+3} x\,dx &= \left.\frac{\cos^{2p+2} x \sin x}{2p+3}\right|_0^{\frac{\pi}{2}} + \frac{2p+2}{2p+3}\int_0^{\frac{\pi}{2}} \cos^{2p+1} x\,dx \\
&= 0 - 0 + \left(\frac{2p+2}{2p+3}\right)\left(\frac{2^{2p}(p!)^2}{(2p+1)!}\right) = \left(\frac{(2p+2)(2p+2)}{(2p+3)(2p+2)}\right)\left(\frac{2^{2p}(p!)^2}{(2p+1)!}\right) \\
&= \frac{2^2(p+1)^2 2^{2p}(p!)^2}{(2p+3)!} = \frac{2^{2p+2}(p+1)!^2}{(2p+3)!}.
\end{aligned}$$

That concludes the induction.

3. Let n be a positive integer. Prove that $\binom{2n}{n}$ is always even.

Answer: We know that $\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n}$. However, symmetry tells us that $\binom{2n-1}{n} = \binom{2n-1}{n-1}$. Therefore, $\binom{2n}{n} = 2\binom{2n-1}{n}$, showing that $\binom{2n}{n}$ is even.