Mathematics 216
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Homework 9
Answers

1. Let n be an integer which is at least 1. Use induction and the previous homework to prove that

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \left(\frac{(2n-1)!}{2^{2n-1}n!(n-1)!}\right)\left(\frac{\pi}{2}\right).$$

Answer: We saw on the last homework that

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{1}{2}\cos x \sin x\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,dx = \left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

When we substitute $n = 1$ into the right-hand side of the formula, we get $(\frac{1!}{2\cdot 1!0!})(\frac{\pi}{2}) = \frac{\pi}{4}$, verifying the formula when $n = 1$.

Now, assuming that

$$\int_0^{\frac{\pi}{2}} \cos^{2k} x\,dx = \left(\frac{(2k-1)!}{2^{2k-1}k!(k-1)!}\right)\left(\frac{\pi}{2}\right),$$

we compute

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2k+2} x\,dx &= \frac{1}{2k+2}\cos^{2k+1} x \sin x\Big|_0^{\frac{\pi}{2}} + \frac{2k+1}{2k+2}\int_0^{\frac{\pi}{2}} \cos^{2k} x\,dx \\ &= 0 + \frac{2k+1}{2k+2}\left(\frac{(2k-1)!}{2^{2k-1}k!(k-1)!}\right)\left(\frac{\pi}{2}\right) \\ &= \left(\frac{2k+1}{2k+2}\right)\left(\frac{2k}{2k}\right)\left(\frac{(2k-1)!}{2^{2k-1}k!(k-1)!}\right)\left(\frac{\pi}{2}\right) \\ &= \frac{(2k+1)!}{2^{2k+1}(k+1)(k)k!(k-1)!} = \frac{(2k+1)!}{2^{2k+1}(k+1)!(k)},\end{aligned}$$

which completes the induction.

2. Find three different complex numbers which solve the equation $z^3 = i$. Express each of the complex numbers in the form $a + bi$, where a and b are real numbers.

Answer: We write $z = \cos\theta + i\sin\theta$, so $z^3 = \cos 3\theta + i\sin 3\theta$, and we need to have $\sin 3\theta = 1$. One possibility is $3\theta = \frac{\pi}{2}$, in which case $\theta = \frac{\pi}{6}$, and $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

Another possibility is $3\theta = \frac{5\pi}{2}$. Then $\theta = \frac{5\pi}{6}$, and $z = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$.

The third possibility is $3\theta = \frac{9\pi}{2}$. Then $\theta = \frac{9\pi}{6} = \frac{3\pi}{2}$, and $z = -i$.

3. Let n be a positive integer, and α any non-negative real number. Prove by induction that

$$(1+\alpha)^n \geq 1 + n\alpha + \frac{n(n-1)}{2}\alpha^2.$$

Be sure in your proof to indicate where you used the fact that $\alpha \geq 0$, because the result is false if α is negative.

Answer: The case $n = 1$ is the inequality

$$(1+\alpha) \geq 1 + \alpha + \frac{1 \cdot 0}{2}\alpha^2 = 1 + \alpha.$$

Therefore, the result is true when $n = 1$.

Assuming now that

$$(1+\alpha)^k \geq 1 + k\alpha + \frac{k(k-1)}{2}\alpha^2,$$

we get

$$\begin{aligned}
(1+\alpha)^{k+1} &= (1+\alpha)(1+\alpha)^k \geq (1+\alpha)\left(1 + k\alpha + \frac{k(k-1)}{2}\alpha^2\right) \\
&= \left(1 + k\alpha + \frac{k(k-1)}{2}\alpha^2\right) + \left(\alpha + k\alpha^2 + \frac{k(k-1)}{2}\alpha^3\right) \\
&= 1 + (k+1)\alpha + \alpha^2\left(\frac{k(k-1)}{2} + k\right) + \frac{k(k-1)}{2}\alpha^3 \\
&\geq 1 + (k+1)\alpha + \alpha^2\left(\frac{k(k-1)}{2} + k\right) \\
&= 1 + (k+1)\alpha + \alpha^2\left(\frac{(k+1)k}{2}\right),
\end{aligned}$$

which establishes the induction.