Mathematics 216
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Homework 10
Answers

1. Define the *harmonic numbers* H_n with the formula

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

Prove that

$$\sum_{k=1}^{n-1} H_k = nH_n - n,$$

if $n \geq 2$.

Answer: We start with the case $n = 2$. The left-hand side of the formula is $H_1 = 1$, and the right hand side is $2H_2 - 2$. We can check that $H_2 = 1 + \frac{1}{2}$, and therefore $2H_2 - 2 = 2(\frac{3}{2}) - 2 = 1$.

Now, we assume that

$$\sum_{k=1}^{p-1} H_k = pH_p - p,$$

and we compute

$$\begin{aligned}\sum_{k=1}^{p} H_k &= \sum_{k=1}^{p-1} H_k + H_p = pH_p - p + H_p = (p+1)H_p - p \\ &= (p+1)\left(H_{p+1} - \frac{1}{p+1}\right) - p = (p+1)H_{p+1} - 1 - p = (p+1)H_{p+1} - (p+1).\end{aligned}$$

That concludes the induction.

2. Define the Fermat numbers f_n with the formula $f_n = 2^{2^n} + 1$ if $n \geq 0$. The first few Fermat numbers are:

$$\begin{aligned} f_0 &= 3 \\ f_1 &= 5 \\ f_2 &= 17 \\ f_3 &= 257 \\ f_4 &= 65537 \\ f_5 &= 4294967297 \end{aligned}$$

Prove that

$$f_0 f_1 f_2 \cdots f_n + 2 = f_{n+1}.$$

Answer: We proceed by induction. In the case $n = 1$, the left-hand side of the formula is $f_0 f_1 + 2 = 3 \cdot 5 + 2 = 17$, and the right-hand side of the formula is $f_2 = 17$.

Assume that $f_0 f_1 f_2 \cdots f_k + 2 = f_{k+1}$, so that $f_0 f_1 f_2 \cdots f_k = f_{k+1} - 2$. We then compute $f_0 f_1 f_2 \cdots f_k f_{k+1} + 2 = (f_{k+1} - 2)f_{k+1} + 2 = (2^{2^{k+1}} - 1)(2^{2^{k+1}} + 1) + 2 = (2^{2^{k+2}} - 1) + 2 = f^{2^{k+2}} + 2$, establishing the induction.

3. Define a sequence of real numbers with the definitions

$$x_1 = 3$$
$$x_n = \sqrt{2x_{n-1} + 1}$$

If n is a positive integer, prove using induction that $x_n \geq x_{n+1}$.
Answer: We compute $x_2 = \sqrt{7} \approx 2.6458$, so we know that $x_1 \geq x_2$. Now, assuming that $x_k \geq x_{k+1}$, we know that

$$2x_k \geq 2x_{k+1}$$
$$2x_k + 1 \geq 2x_{k+1} + 1$$
$$\sqrt{2x_k + 1} \geq \sqrt{2x_{k+1} + 1}$$
$$x_{k+1} \geq x_{k+2}$$

That establishes the induction.