Mathematics 216
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Homework 11
Answers

1. Use the Euclidean algorithm to compute the greatest common divisor d of 3780 and 4342, and find integers m and n so that $d = 3780m + 4342n$.

Answer: We have

$$\begin{aligned}
4342 &= 1 \cdot 3780 + 562 \\
3780 &= 6 \cdot 562 + 408 \\
562 &= 1 \cdot 408 + 154 \\
408 &= 2 \cdot 154 + 100 \\
154 &= 1 \cdot 100 + 54 \\
100 &= 1 \cdot 54 + 46 \\
54 &= 1 \cdot 46 + 8 \\
46 &= 5 \cdot 8 + 6 \\
8 &= 1 \cdot 6 + 2 \\
6 &= 3 \cdot 2
\end{aligned}$$

Therefore, the greatest common divisor is 2, and we have

$$\begin{aligned}
2 &= 1 \cdot 8 + (-1)(6) \\
&= 1 \cdot 8 + (-1)(46 - 5 \cdot 8) \\
&= 6 \cdot 8 + (-1)(46) \\
&= 6 \cdot (54 - 46) + (-1)(46) \\
&= 6 \cdot 54 + (-7)(46) \\
&= 6 \cdot 54 + (-7)(100 - 54) \\
&= 13 \cdot 54 + (-7)(100) \\
&= 13 \cdot (154 - 100) + (-7)(100) \\
&= 13 \cdot 154 + (-20)(100) \\
&= 13 \cdot 154 + (-20)(408 - 2 \cdot 154) \\
&= 53 \cdot 154 + (-20)(408) \\
&= 53 \cdot (562 - 408) + (-20)(408) \\
&= 53 \cdot 562 + (-73)(408) \\
&= 53 \cdot 562 + (-73)(3780 - 6 \cdot 562) \\
&= 491 \cdot 562 + (-73)(3780) \\
&= 491 \cdot (4342 - 3780) + (-73)(3780) \\
&= 491 \cdot 4342 + (-564)(3780)
\end{aligned}$$

One answer is $m = -564$ and $n = 491$.

2. Recall that we defined the Fermat numbers f_n with the formula $f_n = 2^{2^n} + 1$ if $n \geq 0$, and proved that $f_0 f_1 f_2 \cdots f_n + 2 = f_{n+1}$. Use this formula to show that if $m < n$, then f_m and f_n are relatively prime.

Answer: We know from the formula that $f_0 f_1 \cdots f_{n-1} + 2 = f_n$. Because $m < n$, we know that f_m occurs in the product on the left-hand side of the equation. Suppose now that d is a positive number which divides both f_m and f_n. Then the equation says that $d|2$, and so $d = 1$ or $d = 2$. But all of the Fermat numbers are odd, so we cannot have $d = 2$. As a consequence, the only positive divisor of f_m and f_n is 1, showing that $(f_m, f_n) = 1$.