Mathematics 216
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Homework 12
Answers

1. Suppose that c and d are positive integers. Prove using induction that $F_c|F_{cd}$. The formula $F_aF_{b-1} + F_{a+1}F_b = F_{a+b}$ might be helpful.

Answer: We proceed using induction on d. The case $d = 1$ is clear, because the statement $F_c|F_c$ is trivially true.

Assume that $F_c|F_{ck}$. We need to prove that $F_c|F_{c(k+1)}$. Take the given formula, and substitute c for a and ck for b. We get $F_cF_{ck-1} + F_{c+1}F_{ck} = F_{ck+c}$. Now, we know that $F_c|F_cF_{ck-1}$, and by assumption $F_c|F_{c+1}F_{ck}$. Therefore, $F_c|F_{ck+c}$ establishing the induction.

2. Suppose that a and b are positive integers, with $d = (a, b)$. Suppose that we have used the Euclidean algorithm to determine integers x and y so that $ax + by = d$. Prove that x and y are relatively prime.

Answer: We know that $d|a$ and $d|b$. We can take the given equation, and divide through by d, yielding $\frac{a}{d} \cdot x + \frac{b}{d} \cdot y = 1$, where both $\frac{a}{d}$ and $\frac{b}{d}$ are integers. Suppose that $k|x$ and $k|y$, and k is a positive integer. We can conclude that $k|1$, meaning that $k = 1$. Therefore, the only positive integer dividing both x and y is 1, so $(x, y) = 1$.

3. Suppose that n and k are positive integers. Prove that

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}.$$

Answer: Start with the right-hand side of the equation:

$$\frac{n}{k} \cdot \binom{n-1}{k-1} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

4. Let n be a positive integer. Prove that $\binom{3n}{n}$ is always a multiple of 3.

Answer: We have

$$\binom{3n}{n} = \frac{3n}{n} \cdot \binom{3n-1}{n-1} = 3 \cdot \binom{3n-1}{n-1}.$$

We know that $\binom{3n-1}{n-1}$ is an integer, and therefore $\binom{3n}{n}$ is a multiple of 3.