Mathematics 216
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Homework 13
Answers

1. Suppose that a, b, and c are positive integers, where $a|c$, $b|c$, and $(a,b)=1$. Prove that $ab|c$.

Answer: Find integers x and y so that $ax+by=1$. Multiply by c, and we have $acx+bcy=c$. Now, we know that $a|a$ and $b|c$, so $ab|acx$. Similarly, we know that $b|b$ and $a|c$, so $ab|bcy$. Therefore, $ab|c$.

2. Let n be a nonnegative integer, and a any positive real number. Prove that

$$\int_0^1 x^a(\log x)^n\,dx = \frac{(-1)^n n!}{(a+1)^{n+1}}.$$

Answer: We proceed by induction. Because we are given that n is a nonnegative integer, we can begin by checking the equation when $n=0$:

$$\int_0^1 x^a\,dx = \frac{1}{a+1} = \frac{(-1)^0 0!}{(a+1)^{0+1}}.$$

Now, suppose that

$$\int_0^1 x^a(\log x)^k\,dx = \frac{(-1)^k k!}{(a+1)^{k+1}}.$$

We compute the next integral using integration by parts, setting $dv = x^a\,dx$, $v = x^{a+1}/(a+1)$, $u=(\log x)^{k+1}$, and $du = (k+1)(\log x)^k/x\,dx$:

$$\begin{aligned}\int_0^1 x^a(\log x)^{k+1}\,dx &= (\log x)^{k+1}\left(\frac{x^{a+1}}{a+1}\right)\Bigg|_0^1 - \int_0^1 \left(\frac{x^{a+1}}{a+1}\right)(k+1)\left(\frac{(\log x)^k}{x}\right)\,dx\\ &= 0 - \frac{1}{a+1}\lim_{x\to 0^+}(\log x)^{k+1}x^{a+1} - \frac{k+1}{a+1}\int_0^1 x^a(\log x)^k\,dx\\ &= 0-0-\frac{k+1}{a+1}\left(\frac{(-1)^k k!}{(a+1)^{k+1}}\right) = \frac{(-1)^{k+1}(k+1)!}{(a+1)^{k+2}}.\end{aligned}$$

We evaluate

$$\lim_{x\to 0^+}(\log x)^{k+1}x^{a+1} = \lim_{x\to 0^+}(\log x)^{k+1}x \lim_{x\to 0^+} x^a = 0\cdot 0 = 0,$$

using the previously proved fact (homework 5) that $\lim\limits_{x\to 0^+} x(\log x)^n = 0$ for any positive integer n.

3. Suppose that a and b are complex numbers. Prove that

$$|a-b| \ge |a| - |b|$$

by using the triangle inequality.

Answer: The triangle inequality states that $|z+w| \le |z|+|w|$. Substitute $z=a-b$ and $w=b$, and we get $|a| \le |a-b|+|b|$. Subtract $|b|$ from both sides of the inequality, and we have the desired result.