Mathematics 216
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Homework 14
Answers

1. Suppose that a and b are roots of unity. Suppose that $o(a) = m$, $o(b) = n$, and $(m, n) = 1$. Prove that $o(ab) = mn$.

Answer: It is easy to check that $(ab)^{mn} = a^{mn}b^{mn} = (a^m)^n(b^n)^m = 1^n1^m = 1$. That shows that $o(ab) \le mn$. It is harder to show that in fact the order of ab is exactly mn.

Suppose that $(ab)^k = 1$, with $k > 0$. We need to show that $k \ge mn$. Start by taking the equation and raising to the mth power. We get $(ab)^{km} = 1^m = 1$. But $(ab)^{km} = a^{km}b^{km} = (a^m)^k b^{km} = b^{km}$. So we know that $b^{km} = 1$, and hence can conclude that $o(b)|km$. In other words, $n|km$. We are given that $(n, m) = 1$, and therefore we know that $n|k$.

Similarly, take $(ab)^k = 1$ and raise to the nth power. We get $a^{kn}b^{kn} = 1$. Because $b^n = 1$, we know that $a^{kn} = 1$. That tells us that $o(a)|kn$, and because $(m, n) = 1$, we know that $m|k$.

We now have three pieces of information: $m|k$, $n|k$, and $(m, n) = 1$. That is exactly the situation in which a homework problem tells us that $mn|k$, and therefore $k \ge mn$.

2. Find an explicit numerical example in which a and b are roots of unity with $o(a) = m$, $o(b) = n$, $(m, n) \ne 1$, and the order of ab is less than both m and n.

Answer: The simplest example is probably to let $a = e^{2\pi i/6}$, so that $o(a) = 6$, and $b = a^2$, so that $o(b) = 3$. Then $o(ab) = o(a^3) = 2$, and $2 < 3$ and $2 < 6$.

3. Prove that $A \cap B = B$ if and only if $B \subseteq A$.

Answer: Suppose first that $A \cap B = B$. We need to prove that $B \subseteq A$. Let $b \in B$; we need to conclude that $b \in A$. Because $B = A \cap B$, we know that $b \in A \cap B$. By definition, $A \cap B$ consists of elements in both A and B, and therefore $b \in A$.

Now, suppose that $B \subseteq A$. We need to prove that $A \cap B = B$. We know that $A \cap B \subseteq B$, so we need only prove that $B \subseteq A \cap B$. Take an element $b \in B$. Because $B \subseteq A$, we know that $b \in A$. Therefore $b \in A \cap B$. This shows that $B \subseteq A \cap B$.