Mathematics 216
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Homework 15
Answers

1. Suppose that m and k are nonnegative integers. Prove that

$$\int_0^1 x^k(1-x)^m\,dx = \frac{m!k!}{(m+k+1)!}.$$

Answer: Let's do this using induction on m. When $m=0$, and k is arbitrary, we see that

$$\int_0^1 x^k(1-x)^0\,dx = \int_0^1 x^k\,dx = \frac{1}{k+1} = \frac{0!k!}{(k+1)!}.$$

Now, assuming that

$$\int_0^1 x^k(1-x)^p\,dx = \frac{p!k!}{(p+k+1)!}$$

for all non-negative values of k, we use integration by parts, with $u=(1-x)^{p+1}$, $du = -(p+1)(1-x)^p\,dx$, $dv = x^k\,dx$ and $v = x^{k+1}/(k+1)$, to compute

$$\begin{aligned}\int_0^1 x^k(1-x)^{p+1}\,dx &= (1-x)^{p+1}\frac{x^{k+1}}{k+1}\bigg|_0^1 + \int_0^1 (p+1)(1-x)^p\frac{x^{k+1}}{k+1}\,dx \\ &= 0 + \frac{p+1}{k+1}\int_0^1 x^{k+1}(1-x)^p\,dx \\ &= \frac{p+1}{k+1}\cdot\frac{p!(k+1)!}{(p+k+2)!} = \frac{(p+1)p!k!}{(p+k+2)!} = \frac{(p+1)!k!}{(p+k+2)!}.\end{aligned}$$

That concludes the induction.

2. Prove or give a counterexample:

If A, B, and C are sets, then $A\cap(B\setminus C) = (A\cap B)\setminus(A\cap C)$.

Answer: This statement is true. One way to prove it is to use the following identities:

- $B\setminus C = B\cap C^c$.
- de Morgan's Law: $(X\cap Y)^c = X^c\cup Y^c$.
- distributivity: $X\cap(Y\cup Z) = (X\cap Y)\cup(X\cap Z)$.
- $X\cap X^c = \emptyset$.
- $X\cap\emptyset = \emptyset$.
- $X\cup\emptyset = X$.

We have

$$\begin{aligned}(A\cap B)\setminus(A\cap C) &= (A\cap B)\cap(A\cap C)^c = (A\cap B)\cap(A^c\cup C^c) \\ &= [(A\cap B)\cap A^c]\cup(A\cap B\cap C^c) \\ &= [\emptyset\cap B]\cup(A\cap B\cap C^c) = \emptyset\cup(A\cap B\cap C^c) = A\cap(B\cap C^c) \\ &= A\cap(B\setminus C).\end{aligned}$$

3. Suppose that n and k are positive integers, with $k \geq 2$. Prove that

$$(k-1)\binom{nk-1}{n-1} = \binom{nk-1}{n}.$$

Answer: We have

$$\frac{\binom{nk-1}{n}}{\binom{nk-1}{n-1}} = \frac{(nk-1)! \Big/ n!(nk-n-1)!}{(nk-1)! \Big/ (n-1)!(nk-n)!} = \frac{(n-1)!(nk-n)!}{n!(nk-n-1)!}$$

$$= \frac{(n-1)!}{n!} \cdot \frac{(nk-n)!}{(nk-n-1)!} = \frac{1}{n} \cdot \frac{nk-n}{1} = k-1.$$