Mathematics 216
Robert Gross
Homework 16
Answers

1. Prove or give a counterexample:

$$\text{If } A,\ B, \text{ and } C \text{ are sets, then } A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C).$$

Answer: This statement is false. Let $A = B = \{1\}$ and $C = \emptyset$. Then $A \setminus (B \cup C) = \{1\} \setminus \{1\} = \emptyset$, while $(A \setminus B) \cup (A \setminus C) = (\{1\} \setminus \{1\}) \cup (\{1\} \setminus \emptyset) = \emptyset \cup \{1\} = \{1\}$.

2. Suppose that $f : A \to A$ is defined by $f(x) = x^3$, where A is a subset of the complex numbers. Give examples of a set A so that

(a) f is bijective.
(b) f is injective but not surjective.
(c) f is surjective but not injective.
(d) f is neither injective nor surjective.

Answer: (a) Let $A = \{1\}$.

(b) Let $A = \mathbf{Z}$. We know that if m and n are integers and $m \neq n$, then $m^3 \neq n^3$, because this is in fact true if m and n are unequal real numbers. On the other hand, there is no solution in $\mathbf{Z}$ to $f(x) = 2$.

(c) One answer is to let $A = \mathbf{C}$. We know that $f(e^{2\pi i/3}) = 1 = f(1)$, so f is not injective. However, we also know that every complex number has a cube root, so f is surjective.

(d) One answer is $A = \{e^{2\pi i/3}, e^{4\pi i/3}, 1\}$, so that $f(e^{2\pi i/3}) = f(e^{4\pi i/3}) = f(1) = 1$.

3. Show that

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

Answer: Using $X \setminus Y = X \cap Y^c$ and de Morgan's law, we have

$$\begin{aligned}
(A \setminus B) \cup (B \setminus A) &= (A \cap B^c) \cup (B \cap A^c) \\
&= (A \cup B) \cap (A \cup A^c) \cap (B^c \cup B) \cap (B^c \cup A^c) \\
&= (A \cup B) \cap (B^c \cup A^c) \\
&= (A \cup B) \cap (A \cap B)^c \\
&= (A \cup B) \setminus (A \cap B).
\end{aligned}$$