Mathematics 216
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Homework 17
Answers

1. Find sets A and B and functions $f : A \to B$ and $g : B \to A$ so that f is not a bijection, g is not a bijection, and $f \circ g$ is a bijection.

Answer: One possibility is to set $A = \{1, 2\}$ and $B = \{1\}$. Define $g(1) = 1$, and $f(1) = f(2) = 1$. Then f is not an injection, g is not a surjection, but $f \circ g(1) = 1$.

2. Define $f : \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ with the formula $f(x) = (x^2, \sin(x))$.

(*a*) Is f an injection?

(*b*) Is f a surjection?

Answer: The function f is not an injection, because $f(-\pi) = ((-\pi)^2, \sin(-\pi)) = (\pi^2, \sin(\pi)) = f(\pi)$. The function f is not a surjection. There is no solution of $f(x) = (3, 4)$, because there is no solution of $\sin x = 4$ with $x \in \mathbf{R}$.

3. Let $\mu_7 = \{\alpha \in \mathbf{C} : \alpha^7 = 1\}$. Define $f : \mu_7 \to \mu_7$ with the formula $f(\alpha) = \alpha^2$. Is the function f injective, surjective, both, or neither?

Answer: The easiest way to see that f is a bijection is to consider the function $h(\alpha) = \alpha^4$. Then $f(h(\alpha^k)) = f(\alpha^{4k}) = \alpha^{8k} = (\alpha^8)^k = \alpha^k$, and similarly $h(f(\alpha^k)) = h(\alpha^{2k}) = \alpha^{8k} = (\alpha^8)^k = \alpha^k$. Because $f \circ g = \mathrm{id}_{\mu_7}$, and $g \circ f = \mathrm{id}_{\mu_7}$, we know that f is an invertible function. Therefore, f is a bijection.

4. Now set $\mu_{12} = \{\alpha \in \mathbf{C} : \alpha^{12} = 1\}$. Define $g : \mu_{12} \to \mu_{12}$ with the formula $g(\alpha) = \alpha^2$. Is the function g injective, surjective, both, or neither?

Answer: The function g is not an injection, because $g(\alpha^7) = \alpha^{14} = \alpha^{12}\alpha^2 = \alpha^2 = g(\alpha)$. Because the domain and the codomain of g are the same finite set, we know that g cannot be a surjection either.

5. Define a new set operation $A \triangle B$ with the formula

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

On the last homework, we saw that $A \triangle B = (A \cup B) \setminus (A \cap B)$. This operation is sometimes called the *symmetric difference* of the sets A and B.

(*a*) Show that $A \triangle B = B \triangle A$.

(*b*) Show that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Answer: (*a*) We have $A \triangle B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B \triangle A$.

(*b*) This is harder. Start with the observation that $A \triangle B = (A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c = (A \cup B) \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) \cup (B \cap A^c) \cup (B \cap B^c) = \emptyset \cup (A \cap B^c) \cup (B \cap A^c) \cup \emptyset = (A \cap B^c) \cup (B \cap A^c)$. Therefore,

$$\begin{aligned}
(A \triangle B) \triangle C &= \Big((A \cap B^c) \cup (B \cap A^c)\Big) \triangle C \\
&= \Big(\Big((A \cap B^c) \cup (B \cap A^c)\Big)^c \cap C\Big) \cup \Big(\Big((A \cap B^c) \cup (B \cap A^c)\Big) \cap C^c\Big) \\
&= \Big(\Big((A \cap B^c)^c \cap (B \cap A^c)^c\Big) \cap C\Big) \cup \Big(\Big((A \cap B^c) \cup (B \cap A^c)\Big) \cap C^c\Big) \\
&= \Big(\Big((A^c \cup B) \cap (B^c \cup A)\Big) \cap C\Big) \cup \Big((A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c)\Big)
\end{aligned}$$

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\begin{aligned}
&= \Big(\big((A^c \cap B^c) \cup (B \cap A)\big) \cap C\Big) \cup \big((A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c)\big) \\
&= \big((A^c \cap B^c \cap C) \cup (B \cap A \cap C)\big) \cup \big((A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c)\big) \\
&= (A^c \cap B^c \cap C) \cup (A \cap B \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c).
\end{aligned}
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This result is symmetric in A, B, and C, so if instead we compute $A \triangle (B \triangle C) = (B \triangle C) \triangle A$, we must get the same result.