1. Find a function \( f : A \to B \) and subsets \( X, Y \subseteq A \) so that \( f(X \cap Y) \neq f(X) \cap f(Y) \).

Answer: Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is defined by the formula \( f(x) = x^2 \). Let \( X = \{1\} \) and \( Y = \{-1\} \). Then \( X \cap Y = \emptyset \), so \( f(X \cap Y) = f(\emptyset) = \emptyset \). On the other hand, \( f(X) = \{1\} \) and \( f(Y) = \{1\} \), so \( f(X) \cap f(Y) = \{1\} \).

2. Suppose that \( f : A \to B \) is an injective function, and \( X, Y \subseteq A \). Show that \( f(X \cap Y) = f(X) \cap f(Y) \).

Answer: First, pick \( z \in f(X \cap Y) \). By definition, that equation means that \( z = f(x) \) with \( x \in X \cap Y \). Therefore, \( x \in X \) and \( x \in Y \), so \( f(x) \in f(X) \) and \( f(x) \in f(Y) \). In other words, \( z \in f(X) \) and \( z \in f(Y) \), so \( z \in f(X) \cap f(Y) \). Note that we have not used the given information that \( f \) is an injective function, and we have proved that in all circumstances, \( f(X \cap Y) \subseteq f(X) \cap f(Y) \).

Now, suppose that \( z \in f(X) \cap f(Y) \). This means that \( z \in f(X) \) and \( z \in f(Y) \), and therefore \( z = f(x) \) for some \( x \in X \) and \( z = f(y) \) for some \( y \in Y \). But \( f \) is an injection, and the equation \( f(x) = f(y) \) therefore tells us that \( x = y \). In other words, \( z = f(x) \) with \( x \in X \) and \( x \in Y \), so \( x \in X \cap Y \). Therefore, \( z \in f(X \cap Y) \).

3. Let \( A \) be the set of non-negative integers. Define a function \( f : A \times A \to A \) with the formula \( f(a, b) = \binom{a+b}{a-b} \). Is \( f \) an injection? Is \( f \) a surjection? Be sure to explain your answer fully. Remember that we defined \( \binom{r}{s} \) to be 0 if \( s < 0 \), so the function definition makes sense.

Answer: The function \( f \) is not an injection, because \( f(1, 0) = \binom{1}{1} = 1 \) and \( f(2, 0) = \binom{2}{2} = 1 \).

The function \( f \) is also not a surjection, but that is a bit trickier to see. The only binomial coefficient which takes the value 2 is \( \binom{2}{1} \). Hence, solving \( f(a, b) = 2 \) means solving \( \binom{a+b}{a-b} = \binom{2}{1} \), meaning that \( a+b = 2 \) and \( a-b = 1 \). This equation has the unique solution \( a = 1 \frac{1}{2} \) and \( b = 1 \frac{1}{2} \), and those numbers are not in the domain of the function. Therefore, \( f(a, b) = 2 \) has no solution, so the function is not an injection.