Mathematics 216
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Homework 18
Answers

1. Find a function $f : A \to B$ and subsets $X, Y \subset A$ so that $f(X \cap Y) \neq f(X) \cap f(Y)$.
Answer: Suppose that $f : \mathbf{R} \to \mathbf{R}$ is defined by the formula $f(x) = x^2$. Let $X = \{1\}$ and $Y = \{-1\}$. Then $X \cap Y = \emptyset$, so $f(X \cap Y) = f(\emptyset) = \emptyset$. On the other hand, $f(X) = \{1\}$ and $f(Y) = \{1\}$, so $f(X) \cap f(Y) = \{1\}$.

2. Suppose that $f : A \to B$ is an injective function, and $X, Y \subset A$. Show that $f(X \cap Y) = f(X) \cap f(Y)$.
Answer: First, pick $z \in f(X \cap Y)$. By definition, that equation means that $z = f(x)$ with $x \in X \cap Y$. Therefore, $x \in X$ and $x \in Y$, so $f(x) \in f(X)$ and $f(x) \in f(Y)$. In other words, $z \in f(X)$ and $z \in f(Y)$, so $z \in f(X) \cap f(Y)$. Note that we have not used the given information that f is an injective function, and we have proved that in all circumstances, $f(X \cap Y) \subset f(X) \cap f(Y)$.

Now, suppose that $z \in f(X) \cap f(Y)$. This means that $z \in f(X)$ and $z \in f(Y)$, and therefore $z = f(x)$ for some $x \in X$ and $z = f(y)$ for some $y \in Y$. But f is an injection, and the equation $f(x) = f(y)$ therefore tells us that $x = y$. In other words, $z = f(x)$ with $x \in X$ and $x \in Y$, so $x \in X \cap Y$. Therefore, $z \in f(X \cap Y)$.

3. Let A be the set of non-negative integers. Define a function $f : A \times A \to A$ with the formula $f(a, b) = \binom{a+b}{a-b}$. Is f an injection? Is f a surjection? Be sure to explain your answer fully. Remember that we defined $\binom{r}{s}$ to be 0 if $s < 0$, so the function definition makes sense.
Answer: The function f is not an injection, because $f(1, 0) = \binom{1}{1} = 1$ and $f(2, 0) = \binom{2}{2} = 1$.

The function f is also not a surjection, but that is a bit trickier to see. The only binomial coefficient which takes the value 2 is $\binom{2}{1}$. Hence, solving $f(a, b) = 2$ means solving $\binom{a+b}{a-b} = \binom{2}{1}$, meaning that $a + b = 2$ and $a - b = 1$. This equation has the unique solution $a = 1\frac{1}{2}$ and $b = \frac{1}{2}$, and those numbers are not in the domain of the function. Therefore, $f(a, b) = 2$ has no solution, so the function is not an injection.