Mathematics 216
Robert Gross
Homework 19
Answers

1. Suppose that $f : X \to Y$ is a function, and $A \subset X$. Prove or give a counterexample:

$$Y \setminus f(A) \subset f(X \setminus A).$$

As usual, a counterexample means giving explicit sets X, Y, and A, and an explicit function $f : X \to Y$.

Answer: This assertion is false. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2$, and $A = \{1\}$. Then $f(X \setminus A) = f(\mathbf{R} \setminus \{1\}) = \mathbf{R}_{\geq 0}$, the set of nonnegative real numbers. However, $Y \setminus f(A) = \mathbf{R} \setminus \{1\}$, a set which is clearly not a subset of $\mathbf{R}_{\geq 0}$, because $Y \setminus f(A)$ includes all negative real numbers.

2. Suppose that $f : X \to Y$ is a function, and $B \subset Y$.

(a) Show that $f(f^{-1}(B)) \subset B$.

(b) Give an explicit example in which $f(f^{-1}(B)) \neq B$.

(c) Suppose that f is a surjective function. Show that $f(f^{-1}(B)) = B$.

Answer: (a) Suppose that $y \in f(f^{-1}(B))$. That means that $y = f(x)$ for some $x \in f^{-1}(B)$. But if $x \in f^{-1}(B)$, then $f(x) \in B$, which means that $y \in B$. This argument shows that $f(f^{-1}(B)) \subset B$.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$, and let $B = \{-1\}$. Then $f^{-1}(B) = \emptyset$, so $f(f^{-1}(B)) = f(\emptyset) = \emptyset \neq B$.

(c) We already proved in part (a) that $f(f^{-1}(B)) \subset B$, so we need only prove that $B \subset f(f^{-1}(B))$. Suppose that $y \in B$. Because f is surjective, we know that there is some element $x \in X$ so that $f(x) = b$. This means that $x \in f^{-1}(B)$, and then $f(x) \in f(f^{-1}(B))$. In other words, $b \in f(f^{-1}(B))$.