Mathematics 216
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Homework 20
Answers

1. Suppose that $f : X \to Y$, and for every set $A \subseteq X$, $f^{-1}(f(A)) = A$. Prove that f is an injection.

Answer: Suppose that $x_1, x_2 \in X$, and $f(x_1) = f(x_2)$. We need to prove that $x_1 = x_2$.

Let $A = \{x_1\}$. Then $f(A) = \{f(x_1)\}$, and because $f(x_1) = f(x_2)$, we know that $x_1, x_2 \in f^{-1}(\{f(x_1)\})$. We are told that $f^{-1}(f(A)) = A$, and therefore $x_1, x_2 \in \{x_1\}$. That can only happen if $x_1 = x_2$.

2. Let $\mathbf{Q}^\times$ be the set of all non-zero fractions. Define a relation $\sim$ on $\mathbf{Q}^\times$ by saying that $\frac{a}{b} \sim \frac{c}{d}$ if $\frac{ad}{bc} = (\frac{p}{q})^2$, where $\frac{p}{q}$ is a non-zero fraction. For example, $\frac{3}{4} \sim \frac{16}{3}$.

Show that $\sim$ is an equivalence relation.

Answer: We need to show reflexivity, symmetry, and transivity.

- Reflexivity: Because $\frac{ab}{ab} = 1 = (\frac{1}{1})^2$, we have $\frac{a}{b} \sim \frac{a}{b}$.
- Symmetry: If $\frac{a}{b} \sim \frac{c}{d}$, then $\frac{ad}{bc} = (\frac{p}{q})^2$, and so $\frac{bc}{ad} = (\frac{q}{p})^2$, showing that $\frac{c}{d} \sim \frac{a}{b}$. This is one point where it matters that $\frac{p}{q} \neq 0$, so that we can write $\frac{q}{p}$ without fear of division by 0.
- Transivity: If $\frac{a}{b} \sim \frac{c}{d}$, and $\frac{c}{d} \sim \frac{e}{f}$, then $\frac{ad}{bc} = (\frac{p}{q})^2$ and $\frac{cf}{de} = (\frac{r}{s})^2$. Multiplication now yields $\frac{adcf}{bcde} = (\frac{pr}{qs})^2$. Cancellation yields $\frac{af}{be} = (\frac{pr}{qs})^2$, where we know that $\frac{pr}{qs} \neq 0$, because $\frac{p}{q} \neq 0$ and $\frac{r}{s} \neq 0$. This equation shows that $\frac{a}{b} \sim \frac{e}{f}$.

3. Let n be a positive integer. Remember that μ_n, the set of nth roots of unity, is defined by $\mu_n = \{z \in \mathbf{C} : z^n = 1\}$. Remember also that if $z \in \mu_n$, the *order* of z is the smallest positive integer k so that $z^k = 1$.

Define a relation $\sim$ on μ_n by saying that $z \sim w$ if the order of z and the order of w are equal.

(a) Show that this is an equivalence relation.

(b) List the equivalence classes in μ_{10} under this equivalence relation. How many different equivalence classes are there?

Answer: If $z \in \mu_n$, we write $o(z)$ for the order of z.

(a) We need to show reflexivity, symmetry, and transivity.

- Reflexivity: If $z \in \mu_n$, then $o(z) = o(z)$, so $z \sim z$.
- Symmetry: If $z \sim w$, then $o(z) = o(w)$, so $o(w) = o(z)$, and then $w \sim z$.
- Transitivity: If $z \sim s$, and $s \sim w$, then $o(z) = o(s)$ and $o(s) = o(w)$. Therefore $o(z) = o(w)$, and then $z \sim w$.

(b) Let $\zeta = e^{2\pi i/10}$, and then we know that $\mu_{10} = \{\zeta, \zeta^2, \ldots, \zeta^9, \zeta^{10} = 1\}$. We know that $o(\zeta) = 10$, and then our formula $o(\zeta^a) = 10/(a, 10)$ lets us compute the order of each of the other 9 elements of μ_{10}. The possible orders are 1, 2, 5, and 10, and the equivalence classes are:

Order	Element(s)
1	ζ^{10}
2	ζ^5
5	$\zeta^2, \zeta^4, \zeta^6, \zeta^8$
10	$\zeta, \zeta^3, \zeta^7, \zeta^9$