Mathematics 216
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Homework 23
Answers

1. Decide if $f : \mathbf{Z}/7\mathbf{Z} \to \mathbf{Z}/14\mathbf{Z}$ given by the formula $f([x]_7) = [x^2]_{14}$ is a well-defined function. Be sure to explain your answer fully.

Answer: This function is *not* well-defined. We have $[1]_7 = [8]_7$. The formula gives $f([1]_7) = [1]_{14}$, and $f([8]_7) = [64]_{14}$, and $[1]_{14} \neq [64]_{14}$.

2. Let n be a positive integer. Show that $g : \mathbf{Z}/2^n\mathbf{Z} \to \mathbf{Z}/2^{n+1}\mathbf{Z}$ defined by $g([x]_{2^n}) = [x^2]_{2^{n+1}}$ is well-defined.

Answer: Suppose that $[x]_{2^n} = [y]_{2^n}$. We need to prove that $[x^2]_{2^{n+1}} = [y^2]_{2^{n+1}}$. In the language of congruences, we are given $x \equiv y \pmod{2^n}$, and we need to prove that $x^2 \equiv y^2 \pmod{2^{n+1}}$.

If $x \equiv y \pmod{2^n}$, then $y = x + k2^n$, and so $y^2 = (x + k2^n)^2 = x^2 + k2^{n+1} + k^2 2^{2n} = x^2 + 2^{n+1}(k + k^2 2^{n-1}) \equiv x^2 \pmod{2^{n+1}}$. In other words, $[y^2]_{2^{n+1}} = [x^2]_{2^{n+1}}$, which is the desired result.

3. Suppose that A is a finite set, $f : A \to A$, and $g : A \to A$. Suppose in addition that $f \circ g : A \to A$ is a bijection. Prove that f and g are both bijections.

Answer: Suppose that $g(a_1) = g(a_2)$. Then $f(g(a_1)) = f(g(a_2))$. Because $f \circ g$ is a bijection, we can conclude that $a_1 = a_2$. This shows that g must be an injection. But if $g : A \to A$ and A is a finite set, then g must be a bijection.

We could use a similar argument to show that f must be a surjection, and therefore also a bijection. Alternatively, we can reason as follows: g is a bijection, so $g^{-1} : A \to A$ exists and is also a bijection. We are given that $f \circ g$ is a bijection, and therefore $f \circ g \circ g^{-1}$ is a bijection. Because $f \circ g \circ g^{-1} = f$, we conclude that f is a bijection.

4. Give an explicit example to show that the conclusion to the previous problem is *false* if A is an infinite set. You need to tell me what you are using for the set A, what the functions f and g are, and why neither f nor g are bijections.

Answer: We want $f \circ g$ to be a bijection, which requires g to be an injection, and f to be a surjection. We are asked for an example in which neither f nor g are bijections.

One possibility is to take $A = \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ and $f : \mathbf{Z} \to \mathbf{Z}$ to be defined by the formulas:

$$g(n) = \begin{cases} n+1 & n \geq 0 \\ n & n < 0 \end{cases} \qquad f(n) = \begin{cases} n-1 & n \geq 0 \\ n & n < 0 \end{cases}$$

Then $f \circ g(n) = n$. The function g is not a surjection, because there is no solution to $g(n) = 0$, and the function f is not an injection, because $f(0) = f(-1)$. Notice, incidentally, that $g \circ f(0) \neq 0$, even though $f \circ g(0) = 0$.