Mathematics 216
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Homework 24
Answers

1. Consider the function $f : \mathbf{Z}/5\mathbf{Z} \to \mathbf{R}$ defined by the formula $f([n]_5) = \sin(n\pi)$. Is this function well-defined?

Answer: This function *is* well-defined. Suppose that $[n]_5 = [m]_5$. Then $f([m]_5) = \sin(m\pi) = 0$, and $f([n]_5) = \sin(n\pi) = 0$. In other words, the function is identically 0, so it is well-defined.

2. Let n be a positive integer. Show that the function $f : \mathbf{Z}/3^n\mathbf{Z} \to \mathbf{Z}/3^{n+1}\mathbf{Z}$ defined by $f([x]_{3^n}) = [x^3]_{3^{n+1}}$ is well-defined.

Answer: Suppose that $[x]_{3^n} = [y]_{3^n}$, so that $y = x + 3^n k$. Then

$$\begin{aligned} f([y]_{3^n}) = [y^3]_{3^{n+1}} &= [(x+3^n k)^3]_{3^{n+1}} = [x^3 + x^2 3^{n+1} k + x 3^{2n+1} k^2 + 3^{3n} k^3]_{3^{n+1}} \\ &= [x^3 + 3^{n+1}(x^2 k + x 3^n k^2 + 3^{2n-1} k^3)]_{3^{n+1}} = [x^3]_{3^{n+1}}, \end{aligned}$$

showing that the function is well-defined.

3. What is the remainder when 10^{400} is divided by 17?

Answer: We know that $10^{16} \equiv 1 \pmod{17}$. Dividing 400 by 16 yields $400 = 16 * 25$, and so $10^{400} = (10^{16})^2 5 \equiv 1^{25} \equiv 1 \pmod{17}$. Therefore, the remainder is 1.