Mathematics 216
Robert Gross
Homework 25
Answers

1. What is the remainder when 10^{400} is divided by 70?

Answer: We know without effort that $10^{400} \equiv 0 \pmod{10}$. We also know that $10^6 \equiv 0 \pmod{7}$, and $400 = 6 \cdot 66 + 4$. Therefore, $10^{400} = (10^6)^{66} 10^4 \equiv 10^4 \equiv 3^4 \equiv 4 \pmod{7}$.

We now have a (not hard) Chinese Remainder Theorem exercise. Write $10^{400} = 10k$, and we need to solve $10k \equiv 4 \pmod{7}$. We see, either by guesswork or the Euclidean algorithm, that $k \equiv 6 \pmod{7}$, so $k = 7j + 6$, and $10^{400} = 10(7j + 6) = 70j + 60$. Therefore, the answer is 60.

2. What is the remainder when $1^6 + 2^6 + 3^6 + \cdots + 200^6$ is divided by 4?

Answer: Because $1 \equiv 5 \equiv 9 \equiv \cdots \pmod{4}$, and $2 \equiv 6 \equiv 10 \equiv \cdots \pmod{4}$, and $3 \equiv 7 \equiv \cdots \pmod{4}$, we have $1^6 + 2^6 + 3^6 + \cdots + 200^6 \equiv 50 \cdot 1^6 + 50 \cdot 2^6 + 50 \cdot 3^6 \pmod{4}$. Furthermore, $2^6 \equiv 0 \pmod{4}$, and $3^6 \equiv 1 \pmod{4}$, so we have $1^6 + 2^6 + 3^6 + \cdots + 200^6 \equiv 50 + 50 \equiv 0 \pmod{4}$. Hence, the remainder is 0.

3. What is the remainder when $1^6 + 2^6 + 3^6 + \cdots + 200^6$ is divided by 7?

Answer: Now we have $1^6 \equiv 2^6 \equiv \cdots \equiv 6^6 \equiv 1 \pmod{7}$, and $7^7 \equiv 0 \pmod{7}$. Hence, $1^6 + \cdots + 7^6 \equiv 6 \pmod{7}$. Because $200 = 7 \cdot 28 + 4$, we have $1^6 + \cdots + 200^6 \equiv 28(1^6 + \cdots + 7^6) + 1^6 + 2^6 + 3^6 + 4^6 \equiv 1^6 + 2^6 + 3^6 + 4^6 \equiv 1 + 1 + 1 + 1 = 4 \pmod{7}$. In other words, the remainder is 0.

4. What is the remainder when 7^{200} is divided by 33?

Answer: This is a straightforward application of Euler's Theorem. Because $(7, 33) = 1$, we know that $7^{\phi(33)} \equiv 1 \pmod{33}$. Now, $\phi(33) = \phi(3)\phi(11) = 2 \cdot 10 = 20$, so $7^{20} \equiv 1 \pmod{33}$. Therefore, $7200 \equiv (7^{20})^{10} \equiv 1^{10} \equiv 1 \pmod{33}$. Hence, the remainder is 1.