Mathematics 216
Robert Gross
Homework 26
Due April 11, 2012

1. Let $f : X \to Y$ be a function. Suppose that for all subsets $A, B \subset X$, we know that $f(A \cap B) = f(A) \cap f(B)$. Prove or give a counterexample:
   (a) f must be a surjection.
   (b) f must be an injection.

2. Let $M_2(\mathbf{R})$ be the set of all 2×2-matrices with real entries. Define a relation on $M_2(\mathbf{R})$ by saying that the matrices A and B are *similar* if there is an invertible matrix T so that $AT = TB$. Show that similarity of matrices is an equivalence relation.

3. Suppose that n is an integer which is at least 2, a an integer which is relatively prime to n, and $k = o([a]_n)$. Prove that $o([a^d]_n) = k/(k, d)$.

4. Suppose that n is an integer which is at least 2, and a and b are integers which are each relatively prime to n. Suppose that $o([a]_n) = k$, and $o([b]_n) = j$, and $(k, j) = 1$. Prove that $o([ab]_n) = jk$.

5. Suppose that D is an integral domain. Define a relation $\sim$ on $D \times (D \setminus \{0\})$ with the formula $(a, b) \sim (c, d)$ if $ad = bc$. Prove that the relation $\sim$ is transitive.

6. Now define a relation $\sim$ on $\mathbf{Z}/20\mathbf{Z} \times (\mathbf{Z}/20\mathbf{Z} \setminus \{0\})$ with the same formula: $(a, b) \sim (c, d)$ if $ad = bc$. Show that $\sim$ is *not* transitive.