1. Let \( f : X \to Y \) be a function. Suppose that for all subsets \( A, B \subset X \), we know that 
\[ f(A \cap B) = f(A) \cap f(B) \]. Prove or give a counterexample:

(a) \( f \) must be a surjection.

(b) \( f \) must be an injection.

**Answer:** (a) This is false. Consider \( X = \{1\} \), \( Y = \{1, 2\} \), and \( f(1) = 1 \). The function is not surjective, but because the only subsets of \( X \) are \( X \) and \( \emptyset \), we can verify that 
\[ f(A \cap B) = f(A) \cap f(B) \] for all \( A, B \subset X \).

(b) This is true. Suppose that \( f(x_1) = f(x_2) \). We need to prove that \( x_1 = x_2 \). Let \( A = \{x_1\} \) and \( B = \{x_2\} \). We are given \( f(A \cap B) = f(A) \cap f(B) \). We know that \( f(A) \cap f(B) = \{f(x_1)\} \). Therefore, \( A \cap B \neq \emptyset \), because \( f(\emptyset) = \emptyset \). If \( A \cap B \neq \emptyset \), we must have \( A = B \) and then \( x_1 = x_2 \).

2. Let \( M_2(\mathbb{R}) \) be the set of all \( 2 \times 2 \) matrices with real entries. Define a relation on \( M_2(\mathbb{R}) \) by saying that the matrices \( A \) and \( B \) are similar if there is an invertible matrix \( T \) so that 
\[ AT = TB \]. Show that similarity of matrices is an equivalence relation.

**Answer:** We check the usual 3 properties:

- **Reflexivity:** Pick \( A \in M_2(\mathbb{R}) \). The identity matrix \( I \) is invertible, and we know that 
  \[ AI = IA \]. Therefore, \( A \) is similar to itself.

- **Symmetry:** Suppose \( A, B \in M_2(\mathbb{R}) \), with \( A \) similar to \( B \). That means that 
  \[ AT = TB \] for some invertible matrix \( T \). Multiply that equation by \( T^{-1} \) on both the left and right and we get 
  \[ BT^{-1} = T^{-1}A \]. Because \( T^{-1} \) is also an invertible matrix, we conclude that 
  \( B \) is similar to \( A \).

- **Transitivity:** Suppose \( A, B, C \in M_2(\mathbb{R}) \), with \( A \) similar to \( B \) and \( B \) similar to \( C \). This means that there is an invertible matrix \( T \) so that 
  \[ AT = TB \] and another invertible matrix \( S \) so that 
  \[ BS = SC \]. Therefore, \( A(TS) = (AT)S = (TB)S = T(BS) = T(SC) = (TS)C \). Because both \( T \) and \( S \) are invertible, we know that \( TS \) is invertible, and therefore \( A \) is similar to \( C \).

3. Suppose that \( n \) is an integer which is at least 2, \( a \) an integer which is relatively prime to \( n \), and \( k = o([a^n]) \). Prove that \( o([a^d]) = k/(k, d) \).

**Answer:** We know that 
\[ (a^d)^{k/(k,d)} \equiv (a^k)^{d/(k,d)} \equiv 1^{d/(k,d)} \equiv 1 \pmod{n} \], so 
\[ o([a^d]) \leq k/(k, d) \].

Now, suppose that \( (a^d)^j \equiv 1 \pmod{n} \), with \( j > 0 \). We need to show that \( j \geq k/(k, d) \). We have 
\[ a^d \equiv 1 \pmod{n} \], and therefore 
\[ o(a)|dj, \text{ or } k|dj \]. Divide by \( (k, d) \), and we have 
\[ \frac{k}{(k, d)}|\frac{d}{(k, d)}j \]. Now, \( k/(k, d) \) and \( d/(k, d) \) are relatively prime, and therefore we know that 
\[ \frac{k}{(k, d)} \leq j \], which is the desired result.

4. Suppose that \( n \) is an integer which is at least 2, and \( a \) and \( b \) are integers which are each relatively prime to \( n \). Suppose that 
\[ o([a]) = k, \text{ and } o([b]) = j, \text{ and } (k, j) = 1 \]. Prove that 
\[ o([ab]) = kj \].

**Answer:** We know that 
\[ (ab)^{kj} \equiv (a^k)^j(b^j)^k \equiv 1^j1^k \equiv 1 \pmod{n} \]. This shows that 
\[ o(ab) \leq k j \].

Now, suppose that \( m > 0 \) and \( (ab)^m \equiv 1 \pmod{n} \). We need to prove that \( m \geq k j \).
First, raise the equation to the power \( k \), and we get \( a^{km}b^{km} \equiv 1 \pmod{n} \). Because \( a^k \equiv 1 \pmod{n} \), we have \( b^{km} \equiv 1 \pmod{n} \), and hence \( o(b)|km \). Because \( (j,k) = 1 \), we have \( j|m \).

Second, raise the equation to the power \( j \), and we get \( a^{jm}b^{jm} \equiv 1 \pmod{n} \). Because \( b^j \equiv 1 \pmod{n} \), we have \( a^{jm} \equiv 1 \pmod{n} \). Hence \( o(a)|jm \). Because \( (k,j) = 1 \), we have \( k|m \).

Finally, we have \( j|m \), \( k|m \), and \( (j,k) = 1 \), which combine to tell us that \( jk|m \), and hence \( jk \leq m \).

5. Suppose that \( D \) is an integral domain. Define a relation \( \sim \) on \( D \times (D \setminus \{0\}) \) with the formula \((a, b) \sim (c, d)\) if \( ad = bc \). Prove that the relation \( \sim \) is transitive.

Answer: Suppose that \((a, b) \sim (c, d)\) and \((c, d) \sim (e, f)\). That gives \( ad = bc \) and \( cf = de \). Multiply the first equation by \( f \) to get \(adf = bcf \). Multiply the second equation by \( b \) to get \(bcf = bde \). Therefore, \( adf = bde \). Now, because \( d \neq 0 \), we can cancel \( d \) and get \( af = be \), which says that \((a, b) \sim (e, f)\).

6. Now define a relation \( \sim \) on \( \mathbb{Z}/20\mathbb{Z} \times (\mathbb{Z}/20\mathbb{Z} \setminus \{0\}) \) with the same formula: \((a, b) \sim (c, d)\) if \( ad = bc \). Show that \( \sim \) is not transitive.

Answer: We have \((0, 1) \sim (0, 5)\), and \((0, 5) \sim (4, 5)\), but \((0, 1) \not\sim (4, 5)\).