Mathematics 216
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Homework 27
Answers

1. Define a relation $\sim$ on $\mathbf{R}^2$ by setting $(a,b) \sim (c,d)$ if there is a non-zero real number λ so that $(a,b) = (\lambda c, \lambda d)$. Prove that $\sim$ is an equivalence relation. Be sure to explain in your proof where it was important that $\lambda \neq 0$.

Answer: We have $(a,b) = (1 \cdot a, 1 \cdot b)$, so $(a,b) \sim (a,b)$. That takes care of reflexivity. If $(a,b) \sim (c,d)$, then $(a,b) = (\lambda c, \lambda d)$. Then $(c,d) = (\lambda^{-1}a, \lambda^{-1}b)$, so $(c,d) \sim (a,b)$. That takes care of symmetry, and this is one reason why it is important that $\lambda \neq 0$.

Finally, if $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $(a,b) = (\lambda c, \lambda d)$, and $(c,d) = (\mu e, \mu f)$, with $\lambda \neq 0$ and $\mu \neq 0$. Therefore, $\lambda\mu \neq 0$ and $(a,b) = (\lambda\mu e, \lambda\mu f)$, showing that $(a,b) \sim (e,f)$, taking care of transitivity.

2. Suppose that a, b, and c are positive integers, and $(a,b) = 2$ and $(a,c) = 3$. Say as much as possible about (a,bc).

Answer: Because $2|a$ and $3|a$, we know that $6|a$. Similarly, because $2|b$ and $3|c$, we know that $6|bc$. We need to show that no larger integer divides both a and bc.

We can find integers x and y so that $ax + by = 2$, and also find integers m and n so that $am + cn = 3$. Multiplication yields $6 = (ax+by)(am+cn) = a(amx + mby + xcn) + bc(ny)$. Therefore, if $d|a$ and $d|bc$, then $d|6$, and so $d \leq 6$. We have shown that $(a,bc) = 6$.

3. Suppose that a, b, and c are positive integers, and $(a,b) = 2$ and $(a,c) = 4$. Say as much as possible about (a,bc).

Answer: We show first that the only relevant prime in this problem is 2. If p is a prime, and $p|bc$, then $p|b$ or $p|c$. If we also have $p|a$, then we must have $p|4$ or $p|2$, and so we conclude that $p = 2$. In other words, the only prime factor of (a,bc) is 2.

Now, we know that $4|a$ and $4|c$ and $2|b$. We can conclude that $8|bc$. If $8 \nmid a$, then we know that $(a,bc) = 4$. If $8|a$, then at first we can only conclude that $8|(a,bc)$. However, if $16|a$ and $16|bc$, then we could conclude either that $8|c$, in which case $(a,c) \geq 8$, or $4|b$, in which case $(a,b) \geq 4$. Both of those are contradictions, and so we conclude that there are two possibilities: $(a,bc) = 8$ or $(a,bc) = 4$, depending on whether a is a multiple of 8.

4. Suppose that a, b, and c are positive integers, and $(a,b) = 2$ and $(a,c) = 4$. Say as much as possible about (a^2,bc).

Answer: The same analysis shows that the only prime factor of (a^2,bc) is 2. We know that $8|bc$, as before, and we know that $16|a^2$, so we can conclude that $8|(a^2,bc)$. It is possible that $(a^2,bc) = 8$, but it is also possible that $(a^2,bc) > 8$. We cannot have $4|b$, but we can have $8|c$, and in that case $(a^2,bc) = 16$. We cannot have $32|(a^2,bc)$; in that case, we would necessarily have $16|c$ and $8|a$, and then $8|(a,c)$.