Mathematics 216
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Homework 28
Answers

1. Let $f(x) = 2x^2 + 3x + 1$ and let $g(x) = 3x^4 + 2x + 1$. Consider both $f(x)$ and $g(x)$ as elements of $\mathbf{F}_5[x]$, and compute $q(x)$ and $r(x)$ so that $g(x) = q(x)f(x) + r(x)$ with $\deg(r) < 2$.

Answer: We have

$$
\begin{array}{r|l}
 & 4x^2 + 4x + 2 \\ \hline
2x^2 + 3x + 1 & 3x^4 + 0x^3 + 0x^2 + 2x + 1 \\
 & \underline{3x^4 + 2x^3 + 4x^2} \\
 & \quad 3x^3 + x^2 + 2x + 1 \\
 & \quad \underline{3x^3 + 2x^2 + 4x} \\
 & \qquad 4x^2 + 3x + 1 \\
 & \qquad \underline{4x^2 + x + 2} \\
 & \qquad\quad 2x + 4
\end{array}
$$

and so

$$3x^4 + 2x + 1 = (4x^2 + 4x + 2)(2x^2 + 3x + 1) + (2x + 4).$$

2. Find the greatest common divisor of $x^5 - 1$ and $2x^2 + 3x + 1$ as elements of $\mathbf{F}_{11}[x]$. Then find polynomials $f, g \in \mathbf{F}_{11}[x]$ so that $(x^5 - 1)f + (2x^2 + 3x + 1)g = (x^5 - 1, 2x^2 + 3x + 1)$. Remember that the greatest common divisor is defined to be monic.

Answer: We have

$$
\begin{array}{r|l}
 & 6x^3 + 2x^2 + 5x + 8 \\ \hline
2x^2 + 3x + 1 & x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 10 \\
 & \underline{x^5 + 7x^4 + 6x^3} \\
 & \quad 4x^4 + 5x^3 + 0x^2 + 0x + 10 \\
 & \quad \underline{4x^4 + 6x^3 + 2x^2} \\
 & \qquad 10x^3 + 9x^2 + 0x + 10 \\
 & \qquad \underline{10x^3 + 4x^2 + 5x} \\
 & \qquad\quad 5x^2 + 6x + 10 \\
 & \qquad\quad \underline{5x^2 + 2x + 8} \\
 & \qquad\qquad 4x + 2
\end{array}
$$

So

$$x^5 - 1 = (6x^3 + 2x^2 + 5x + 8)(2x^2 + 3x + 1) + (4x + 2)$$

Now

$$
\begin{array}{r|l}
 & 6x + 6 \\ \hline
4x + 2 & 2x^2 + 3x + 1 \\
 & \underline{2x^2 + x} \\
 & \quad 2x + 1 \\
 & \quad \underline{2x + 1} \\
 & \qquad 0
\end{array}
$$

So

$$2x^2 + 3x + 1 = (6x + 6)(4x + 2)$$

Therefore, the greatest common divisor is $4x + 2$, except that $4x + 2$ is not monic. We have to multiply by $4^{-1} = 3$:

$$4x + 2 = (1)(x^5 - 1) + (-6x^3 - 2x^2 - 5x - 8)(2x^2 + 3x + 1)$$
$$x + 6 = (3)(x^5 - 1) + (4x^3 + 5x^2 + 7x + 9)(2x^2 + 3x + 1)$$

The greatest common divisor is $x + 6$.

3. On a previous homework, we defined the concept of similar matrices: $A, B \in M_2(\mathbf{R})$ are similar if there is an invertible matrix T so that $AT = TB$. Suppose that A and B are similar and that A is invertible. Prove that B is invertible.

Answer: Suppose that A and B are similar. Then there is an invertible matrix T so that $AT = TB$. Therefore, $\det(AT) = \det(TB)$. We know that $\det(XY) = \det(X)\det(Y)$, and so we have $\det(A)\det(T) = \det(T)\det(B)$. Because T is invertible, we know that $\det(T) \neq 0$, and therefore we can cancel and get $\det(A) = \det(B)$. Now, if A is invertible, $\det(A) \neq 0$, so $\det(B) \neq 0$, and therefore B is invertible.

4. Suppose that $f : \mathbf{Q} \to \mathbf{Q}$ is defined by $f(x) = \dfrac{x}{x^2 - 2}$. Is f a surjection? Is f an injection? Be sure to explain your answer.

Answer: To see if f is a surjection, we can pick a specific value y in the codomain and see if we can solve $f(x) = y$. In particular, suppose we try to solve $f(x) = 10$. We get $x/(x^2 - 2) = 10$, or $x = 10x^2 - 20$, or $10x^2 - x - 20 = 0$. The discriminant of this equation is 801, and because 801 is not a perfect square, we know that the roots are not rational numbers. Therefore, the function is not a surjection.

To see if f is an injection, we set $f(a) = f(b)$, and see if we can conclude that $a = b$. We have

$$\begin{aligned} f(a) &= f(b) \\ \frac{a}{a^2 - 2} &= \frac{b}{b^2 - 2} \\ ab^2 - 2a &= a^2 b - 2b \\ ab^2 - a^2 b &= 2(a - b) \\ ab(b - a) &= -2(b - a) \\ ab &= -2 \end{aligned}$$

So we see that if $ab = -2$, then it is possible for $f(a)$ and $f(b)$ to be equal. In particular, we have $f(1) = -1$ and $f(-2) = -1$, so the function is not injective.

5. Suppose that $g : \mathbf{Z} \to \mathbf{Q}$ is defined by $g(x) = \dfrac{x}{x^2 - 2}$. Is g a surjection? Is g an injection? Be sure to explain your answer.

Answer: Precisely the same reasoning as in the previous question applies: the function is neither surjective nor injective.