Mathematics 216
Robert Gross
Homework 29
Answers

1. Factor $x^4 + 1$ into irreducible factors in $\mathbf{Q}[x]$.

Answer: We know that $c^4 + 1$ is always positive if $c \in \mathbf{Q}$, and therefore $x^4 + 1$ has no linear factor $x - c$.

Now, suppose that $x^4 + 1 = (x^2 + ax + b)(x^2 + cx + d)$. We compute that $(x^2 + ax + b)(x^2 + cx + d) = x^4 + (a + c)x^3 + (ac + b + d)x^2 + (bc + ad)x + bd$, leading to the four equations:

$$a + c = 0 \tag{1}$$
$$ac + b + d = 0 \tag{2}$$
$$bc + ad = 0 \tag{3}$$
$$bd = 1 \tag{4}$$

Take $c = -a$ from (1), and substitute into (3) to get $a(-b + d) = 0$. The possibilities are $a = 0$ or $b = d$.

If $a = 0$, then (1) says that $c = 0$. Then (2) implies that $b = -d$. Substitution into (4) means that $-b^2 = 1$, which has no solution in $\mathbf{Q}$.

If $b = d$, then (4) means that $b^2 = 1$, and so $b = \pm 1$. Consider first $b = -1$. Substitution into (2) implies $-a^2 - 2 = 0$, or $a^2 = -2$, which has no solution in $\mathbf{Q}$.

Finally, if $b = 1$, we have $a^2 = 2$, which again has no solution in $\mathbf{Q}$.

2. Factor $x^4 + 1$ into irreducible factors in $\mathbf{R}[x]$.

Answer: The same analysis as in the previous question works, up until the final paragraph. Now the equation $a^2 = 2$ indeed has solutions. Set $a = \sqrt{2}$, and then $c = -\sqrt{2}$, and $b = d = 1$. We see that $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$.

3. Factor $x^4 + 1$ into irreducible factors in $\mathbf{C}[x]$.

Answer: Finding the four roots of $x^4 + 1$ is the same as solving the equation $x^4 + 1 = 0$, or $x^4 = -1 = e^{\pi i} = e^{3\pi i} = e^{5\pi i} = e^{7\pi i}$. The roots are

$$\begin{aligned} x = e^{\pi i/4} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\ x = e^{3\pi i/4} &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\ x = e^{5\pi i/4} &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \\ x = e^{7\pi i/4} &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{aligned}$$

and so

$$x^4 + 1 = \left(x - (\tfrac{\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2})\right)\left(x - (\tfrac{-\sqrt{2}}{2} + i\tfrac{\sqrt{2}}{2})\right)\left(x - (\tfrac{-\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2})\right)\left(x - (\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2})\right)$$

4. Factor $x^4 + 1$ into irreducible factors in $\mathbf{Z}/2\mathbf{Z}[x]$.

Answer: This one is done by trial and error, and we end up with $x^4 + 1 = (x + 1)^4$ in $\mathbf{F}_2[x]$.