Mathematics 216
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Homework 31
Answers

1. Suppose that $f(x), g(x) \in \mathbf{C}[x]$ are two monic polynomials, with $\deg(f) = \deg(g) = n \geq 1$. Suppose also that $f(1) = g(1)$, $f(2) = g(2)$, $\ldots$, $f(n) = g(n)$. Show that $f(x) = g(x)$. *Hint:* Let $h(x) = f(x) - g(x)$. What is the degree of h? What are some of its roots?

Answer: Let $h(x) = f(x) - g(x)$. Because $f(x)$ and $g(x)$ are both monic, we know that either $h(x) = 0$ or $\deg(h) < n$. We also know that $h(1) = h(2) = \cdots = n(n) = 0$, so $h(x)$ has at least n roots. The only conclusion is that $h(x) = 0$, which says that $f(x) = g(x)$.

2. Suppose that we remove the assumption that $f(x)$ and $g(x)$ are monic in the previous problem. Show by example that we can no longer conclude that $f(x) = g(x)$.

Answer: Suppose that $f(x) = x^2 - 3x + 2$ and $g(x) = 2x^2 - 6x + 4$. Then $f(1) = f(2) = 0$, and $g(1) = g(2) = 0$. But $f(x) \neq g(x)$.

3. Factor $x^4 + x^2 + 1$ into irreducible factors in $\mathbf{Q}[x]$.

Answer: We know that $f(x) = x^4 + x^2 + 1$ has no roots in $\mathbf{Q}$ because we know that $f(a) > 0$ for every $a \in \mathbf{Q}$. The only possible factorization is $x^4 + x^2 + 1 = (x^2 + ax + b)(x^2 + cx + d)$, and we can assume that $a, b, c, d \in \mathbf{Z}$. Equating coefficients yields

$$\begin{aligned} a + c &= 0 \\ ac + b + d &= 1 \\ ad + bc &= 0 \\ bd &= 1 \end{aligned}$$

The final equation says that either $b = d = 1$ or $b = d = -1$. Try $b = d = -1$ first, and then we have $ac = 3$ and $a + c = 0$, and those two equations are contradictory. Try $b = d = 1$, and we have $ac = -1$ and $a + c = 0$, with solution $a = 1$ and $c = -1$. We can now see that $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$.

4. Factor $x^4 + x^2 + 1$ into irreducible factors in $\mathbf{F}_2[x]$.

Answer: Let $f(x) = x^4 + x^2 + 1$. We see that $f(0) = 1$ and $f(1) = 1$, so there are no linear factors. We also know that $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1) = (x^2 + x + 1)^2$, and that is as far as the polynomial can be factored.

5. Factor $x^4 + x^2 + 1$ into irreducible factors in $\mathbf{F}_7[x]$.

Answer: Again, we have $x^4 + x^2 + 1 = (x^2 - x + 1)(x^2 + x + 1)$. Now, however, the quadratic polynomials factor further. The first factor has roots 3 and 5, and the second factor has roots 2 and 4. Hence, $x^4 + x^2 + 1 = (x - 3)(x - 5)(x - 2)(x - 4)$.