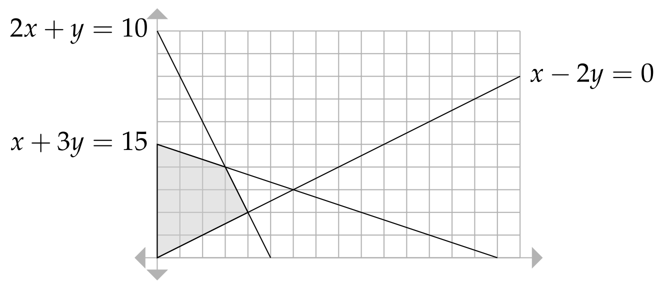
Mathematics 235
Examination 1
February 18, 2011

1. *(20 points)* Consider the following linear programming problem:
Maximize $5x + 3y$ subject to:

$$\begin{aligned} x + 3y &\le 15 \\ x - 2y &\le 0 \\ 2x + y &\le 10 \\ x, y &\ge 0 \end{aligned}$$

a. Graph the feasible region for this problem. Find the coordinates of all of the corner points of the region.
b. Find the optimal solution to this problem, using either the corner point method or graphing an isoprofit line. State the values of x, y, and the objective function.

Answer: The graph looks like:



The corner points are $(0, 0)$, $(0, 5)$, $(3, 4)$, and $(4, 2)$. The value of the objective function at each of these points is:

x	0	0	3	4
y	0	5	4	2
$5x + 3y$	0	15	27	26

The optimal solution occurs when $x = 3$ and $y = 4$, and the value of the objective function is 27.

2. *(20 points)* **Rob's Select Whiskey Company** sells two different types of whiskey: **regular** and **premium**. Both are manufactured by blending raw alcohol, barrel-aged whiskey, flavoring extract, and caramel coloring, and both are sold in 750ml bottles.

A bottle of regular whiskey consists of 500ml of alcohol, 240ml of aged whiskey, 8ml of caramel, and 2ml of flavoring extract. A bottle of premium whiskey consists of 100ml of alcohol, 645 ml of aged whiskey, 1ml of caramel, and 4ml of flavoring extract. The profit on a bottle of regular whiskey is $3.45. The profit on a bottle of premium whiskey is $4.80.

There are 40L of raw alcohol, 50L of barrel-aged whiskey, 10L of caramel, and 2L of flavoring extract available. (There are 1000ml in 1L.)

Formulate a linear programming model to help Rob's Select maximize profit. Your model should clearly define all decision variables, state the objective function, and include all relevant constraints.

Answer: Let

$$\begin{aligned} r &= \text{number of bottles of regular manufactured} \\ p &= \text{number of bottles of premium manufactured} \end{aligned}$$

We need to maximize $3.45r + 4.80p$ given

Raw alcohol:	$500r + 100p \le 40 \cdot 1000$
Aged whiskey:	$240r + 645p \le 50 \cdot 1000$
Caramel:	$8r + p \le 10 \cdot 1000$
Flavoring extract:	$2r + 4p \le 2 \cdot 1000$

3. *(20 points)* **Giselle's Perfumerie** sells perfume, bath water, and *eau de cologne* by blending together alcohol and scents. Perfume contains is 90% alcohol and 10% scents. Bath water is 85% alcohol and 15% scents. *Eau de cologne* is 80% alcohol and 20% scents. The respective profits on the three products are $1.20, $1.10, and $0.95 per 1L bottle. Giselle's has on hand right now 5,000L of alcohol and 750ml of scent. *Note:* 1L=1000ml.

Here is an incomplete Excel spreadsheet that contains my partial solution:

	A	B	C	D
1	**Giselle's**			
2				
3		Alcohol	Scents	Profit
4	Perfume	0.90	0.10	
5	Bath water		0.15	1.10
6	Eau de cologne		0.20	0.95
7				
8		Amount		
9	Perfume			
10	Bath water			
11	Eau de cologne			
12				Available
13	Alcohol		<=	5000
14	Scent		<=	
15				
16				
17	Profit:			

Your job is to finish the spreadsheet by filling in the missing entries. If you think that the cells should not have any entries, write the words "No entry" in the cell. No credit will be given for any cells left completely blank.

Answer:

Cell	Entry
D4	`1.20`
B5	`0.85`
B6	`0.80`
B9	`No entry (Solver will compute)`
B10	`No entry (Solver will compute)`
B11	`No entry (Solver will compute)`
B13	`=SUMPRODUCT(B4:B6,B9:B11)`
B14	`=SUMPRODUCT(C4:C6,B9:B11)`
D14	`0.750`
B17	`=SUMPRODUCT(D4:D6,B9:B11)`

4. *(20 points)* **Corazzo's** manufactures pork sausage in 2,000-pound batches by blending pork, beef, and filler. The cost of pork is \$2.50/lb, the cost of beef is \$1.80/lb, and the cost of filler is \$1.00/lb. Each batch must contain:

1. At least 800 pounds of pork
2. At least 300 pounds of beef
3. No more than 30% filler.

I wrote an *Excel* spreadsheet to solve the problem of making a batch of sausage at minimal cost. The resulting sensitivity report is:

Variable Cells

Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
Pork	800	0	2.50	1E+30	0.7
Beef	600	0	1.80	0.7	0.8
Filler	600	0	1.00	0.8	1E+30

Constraints

Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
Beef minimum	600	0.00	300	300	1E+30
Pork minimum	800	0.70	800	300	800
Filler maximum	600	-0.80	600	300	600
Batch size	2000	1.56	2000	1E+30	428.5714

Based on this sensitivity report, write your answers to the following questions on the facing page or the back of this page. You may not be able to answer all of these questions with the information available; in that case, indicate that no answer is possible.

a. What is the optimal solution to this problem? In other words, how should Corazzo's put together a 2,000-pound batch of sausage in order to minimize costs, and what is the resulting cost of the batch of sausage?

b. Which constraints are binding?

Each of the following questions poses an independent "what-if" scenario.

c. If the minimum amount of beef in each batch changed to 200 pounds, say as much as possible about what would happen to the cost of a batch and the optimal solution.

d. If the minimum amount of pork in each batch changed to 1,000 pounds, say as much as possible about what would happen to the cost of a batch and the optimal solution.

e. Suppose that the cost of a pound of pork changed to \$2/lb. Say as much as possible about what would happen to the cost of a batch of sausage and the optimal solution.

f. Suppose that the cost of a pound of beef changed to \$1.50/lb. Say as much as possible about what would happen to the cost and the optimal solution.

g. Suppose that the batch size decreased to 1,500 pounds, without changing the minimum requirements for pork and beef or the maximum requirement for filler. What would be the new minimal cost of a batch of sausage, and what happens to the optimal solution?

h. Suppose that the batch size increased to 2,500 pounds, without changing the minimum requirements for pork and beef or the maximum requirement for filler. What would be the new minimal cost of a batch of sausage, and what happens to the optimal solution?

Answer: (*a*) The optimal solution is to make a batch of sausage from 800lb of pork, 600lb of beef, and 600lb of filler, for a total cost of \$3680.

(*b*) The binding constraints are the pork minimum of 800lb, the filler maximum of 30%, and (of course) the constraint that each batch must consist of 2000lb.

(*c*) Decreasing the minimum amount of beef to 200lb is within the allowable decrease. Because this is a surplus constraint, the solution will not change, nor will the cost.

(*d*) Increasing the minimum amount of pork to 1000lb is within the allowable increase. The optimal solution will change unpredictably (though we of course know that it will include at least 1000lb of pork), and the cost will increase by $200 \cdot 0.70 = \$140$.

(*e*) Changing the cost of pork to \$2/lb is within the allowable decrease. The optimal solution will not change, and the cost of a batch of sausage will decrease to \$3280.

(*f*) Changing the cost of beef to \$1.50/lb is within the allowable decrease. The optimal solution will not change, and the cost of a batch of sausage will decrease to \$3500.

(*g*) A change in the batch size to 1500lb is outside of the allowable decrease on batch size. No prediction can be made about the optimal solution or cost, though clearly the cost will decrease.

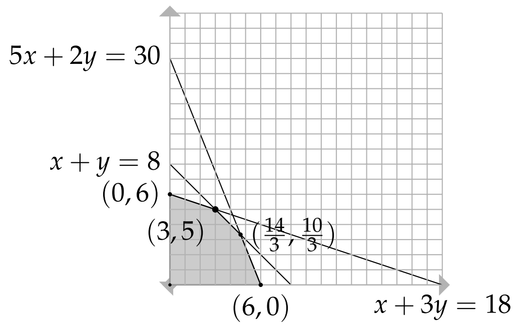
(*h*) A change in the batch size to 2500lb is within the allowable increase. The optimal solution will change unpredictably, and the cost of a batch will increase by $500 \cdot 1.56 = \$780$.

5. *(20 points)* Consider the following linear programming problem:

Maximize $4x + 7y$ subject to:

$$\begin{aligned} 5x + 2y &\leq 30 \\ x + y &\leq 8 \\ x + 3y &\leq 18 \\ x, y &\geq 0 \end{aligned}$$

When I solved the problem, I found that the optimal solution occurs when $x = 3$ and $y = 5$.



a. Suppose that the coefficient of y in the objective function changes. If we write the objective function as $4x + Cy$, for what range of values of C will the optimal solution remain $x = 3$ and $y = 5$?

b. What is the shadow price of the constraint $x + 3y \le 18$? For what range of values on the right-hand side of the constraint is this shadow price valid?

Answer: (*a*) The binding constraints are $x + 3y \le 18$, with slope $-\frac{1}{3}$ and $x + y \le 8$, with slope -1. The slope of an isoprofit line of the form $4x + Cy = K$ is $-\frac{4}{C}$. We therefore have

$$-1 \le -\frac{4}{C} \le -\frac{1}{3}$$
$$1 \ge \frac{4}{C} \ge \frac{1}{3}$$
$$1 \le \frac{C}{4} \le 3$$
$$4 \le C \le 12$$

If you prefer, you can say that the coefficient of y has an allowable decrease of 3 and an allowable increase of 5.

(*b*) We set $x + 3y = 18 + \lambda$ and solve simultaneously with $x + y = 8$. Subtraction yields $2y = 10 + \lambda$, or $y = 5 + \frac{1}{2}\lambda$. We have $x = 8 - y = 8 - (5 + \frac{1}{2}\lambda) = 3 - \frac{1}{2}\lambda$. Finally, we compute $4x + 7y$ and get $4(3 - \frac{1}{2}\lambda) + 7(5 + \frac{1}{2}\lambda) = 47 + \frac{3}{2}\lambda$, so the shadow price of the constraint is $\frac{3}{2}$.

To find the feasible range, we first use $x \ge 0$, implying $3 - \frac{1}{2}\lambda \ge 0$, or $6 \ge \lambda$. To get the lower bound, we substitute into $5x + 2y \le 30$, getting $5(3 - \frac{1}{2}\lambda) + 2(5 + \frac{1}{2}\lambda) \le 30$, implying $25 - \frac{3}{2}\lambda \le 30$. We then get $-\frac{3}{2}\lambda \le 5$, meaning that $\lambda \ge -\frac{10}{3} \approx -3.333$.

Grade	Number of people
97	2
95	1
93	1
91	1
90	1
87	1
86	3
84	1
83	1
82	2
80	1
77	2
76	1
73	1
72	1
71	1
69	1
65	1
59	1

Mean: 81.58
Standard deviation: 9.87