Mathematics 235
Examination 2
April 1, 2011
Answers

1. (*25 points*) **Ahab's Tea Company** has pension payments to make at the start of each of the next few years:

Year	1	2	3	4
Payment (thousands of dollars)	0	141	131	120

Note that no money is due at the start of the current year.

Ahab can choose from two investments, both of which are only available right now:

Security	**Current Price**	**Rate (%)**	**Years to Maturity**
A	\$1,025	5.2	2
B	\$1,015	4.5	3

The par value of each of the bonds is \$1,000. Ahab can also put money into a savings account, which pays 1% annually. *Bonds may not be purchased in fractional amounts.*

Formulate an integer linear program to minimize the amount of money which Ahab must set aside (including bond purchases) to meet these obligations. *Do not try to solve the problem.*

Answer: Let

$$\begin{aligned} A &= \text{number of bonds of type } A \text{ which are purchased} \\ B &= \text{number of bonds of type } B \text{ which are purchased} \\ S_k &= \text{amount in savings in year } k \end{aligned}$$

All dollar amounts are in thousands of dollars.

Our objective is to minimize S_0, given that A and B are positive integer variables, $S_k \geq 0$, and

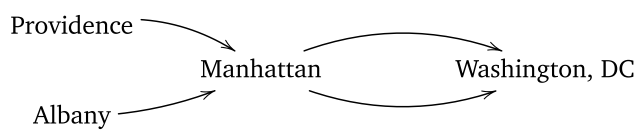
Year 1: $$S_0 - 1.025A - 1.015B = S_1$$

Year 2: $$1.01S_1 + 0.052A + 0.045B - 141 = S_2$$

Year 3: $$1.01S_2 + 0.052A + 0.045B + A - 131 = S_3$$

Year 4: $$1.01S_3 + 0.045B + B - 120 \geq 0$$

2. (*25 points*) **Deathwish Transportation** runs a bus each morning from Providence to Manhattan, and another one from Albany to Manhattan. Both of the vehicles then continue to Washington, DC:



Each of the two vehicles holds up to 80 passengers. The demand and price for the tickets are:

Route	*Demand*	*Ticket Price*
Providence–Manhattan	57	\$25
Albany–Manhattan	43	\$22
Providence–Washington	44	\$45
Albany–Washington	46	\$45
Manhattan–Washington	50	\$30

Deathwish will not sell fractional tickets, and wishes to maximize ticket revenue.

Formulate a linear program to solve this problem. Be sure to tell me what each of your variables stands for, and mention whether any of them are integer-valued or binary. List all of the constraints, and indicate the objective function clearly. *Do not try to solve the problem after you formulate it.*

Answer: Let

$$\begin{aligned} x_{PM} &= \text{number of Providence–Manhattan tickets sold} \\ x_{AM} &= \text{number of Albany–Manhattan tickets sold} \\ x_{PW} &= \text{number of Providence–Washington tickets sold} \\ x_{AW} &= \text{number of Albany–Washington tickets sold} \\ x_{MW} &= \text{number of Manhattan–Washington tickets sold} \end{aligned}$$

We wish to maximize $25x_{PM} + 22x_{AM} + 45x_{PW} + 45x_{AW} + 30x_{MW}$ given

Originate in Providence:	$x_{PM} + x_{PW} \leq 80$
Originate in Albany:	$x_{AM} + x_{AW} \leq 80$
End in Washington:	$x_{PW} + x_{AW} + x_{MW} \leq 160$
Demand constraints:	$x_{PM} \leq 57$
	$x_{AM} \leq 43$
	$x_{PW} \leq 44$
	$x_{AW} \leq 46$
	$x_{MW} \leq 50$

and all variables are nonnegative. It is in fact not necessary to specify that these variables are integer-valued, because this is a transportation problem, but it is not incorrect to do so.

3. (*25 points*) **Jill's Candy Company** manufactures popcorn, cotton candy, and peanuts. The profit per box and weekly demand for each of these is given in the chart, along with the set-up cost to manufacture each of the three candies:

	Profit/box	Demand	Set-up cost
Popcorn	\$2.34	1200	\$250
Cotton candy	\$1.90	1500	\$200
Peanuts	\$1.75	1600	\$150

Remember that the set-up cost is the fixed cost to set up the production line to manufacture a particular item.

Jill can manufacture a total of 2500 boxes each week, so not all of the demands can be met. Jill cannot manufacture any fractional boxes of candy.

Formulate up a linear programming problem that tells how many boxes of popcorn, cotton candy, and peanuts Jill should manufacture in order to meet as much demand as possible and maximize her net profit (that is, the difference between her profit and her set-up costs). Be sure to tell me what each of your variables stands for, and mention whether any of them are integer-valued or binary. List all of the constraints, and indicate the objective function clearly. *Do not try to solve the problem after you formulate it.*

Answer: Let

$$\begin{aligned}
r &= \text{number of boxes of popcorn manufactured} \\
t &= \text{number of boxes of cotton candy manufactured} \\
p &= \text{number of boxes of peanuts manufactured} \\
y_r &= \begin{cases} 1 & \text{if popcorn is manufactured} \\ 0 & \text{if popcorn is not manufactured} \end{cases} \\
y_t &= \begin{cases} 1 & \text{if cotton candy is manufactured} \\ 0 & \text{if cotton candy is not manufactured} \end{cases} \\
y_p &= \begin{cases} 1 & \text{if peanuts are manufactured} \\ 0 & \text{if peanuts are not manufactured} \end{cases}
\end{aligned}$$

We need to maximize $2.35r + 1.90t + 1.75p - 250y_r - 200y_t - 150y_p$ given

Total demand:	$r + t + p \leq 2500$
Turn on y_r:	$r \leq 1200y_r$
Turn off y_r:	$y_r \leq r$
Turn on y_t:	$t \leq 1500y_t$
Turn off y_t:	$y_t \leq t$
Turn on y_p:	$p \leq 1600y_p$
Turn off y_p:	$y_p \leq p$

where r, t, and p are integer variables, y_r, y_t, and y_p are binary variables, and all are nonnegative.

4. (*25 points*) **Eden Apples** manufactures two products: applesauce and apple juice. The cost of manufacturing a jar of applesauce is \$0.60, and the cost of manufacturing a jar of apple juice is \$0.85. Eden sells each jar of applesauce for \$1.45, and each jar of apple juice for \$1.75.

Without any advertising, the demand for applesauce is 5,000 jars, and the demand for apple juice is 4,000 jars. Eden advertises each product separately. Each dollar spent on advertising applesauce increases sales by 3 jars, and each dollar spent on advertising apple juice increases sales by 5 jars. The advertising budget is \$16,000. Eden must spend a minimum of \$5,000 advertising each product.

I solved this problem using *Excel* and *Solver*, and here is a slightly modified version of the sensitivity report:

Variable Cells

Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
Applesauce	20000	0	0.85	0.65	0.85
Juice	59000	0	0.90	1E+30	0.39
Advertising: Applesauce	5000	0	-1	1.95	1E+30
Advertising: Juice	11000	0	-1	1E+30	1.95

Constraints

Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
Applesauce Adver. Min.	5000	-1.95	5000	6000	5000
Juice Adver. Min.	11000	0	5000	6000	1E+30
Advertising budget	16000	3.50	16000	1E+30	16000
Applesauce demand	20000	0.85	20000	1E+30	20000
Juice demand	59000	0.90	59000	1E+30	59000

Use only the sensitivity report to answer these question. Write your answers to the following questions on the facing page or the back of this page. **No credit will be given for answers without explanations.** You may not be able to answer all of these questions with the information available; in that case, indicate that no answer is possible.

a. What is the optimal solution to this problem? In other words, how many jars of each product should Eden make, and how much should Eden spend advertising each product?

b. What is Eden's revenue?

Each of the following questions poses an independent "what—if" scenario.

c. Suppose that Eden had \$4,000 additional dollars to spend on advertising. Should they be spent? If so, what will happen to the profit and optimal solution?

d. Suppose that Eden removed the advertising minimum of \$5,000 for each product. Can you tell from the sensitivity report what would happen to the profit and the optimal solution? If so, what is the answer?

e. Suppose that Eden decreases the price of a jar of applesauce to \$1.25. Can you tell what will happen to the optimal solution and the profit?

f. Suppose that a temporary manufacturing problem increases Eden's cost of manufacturing a jar of apple juice from \$0.85 to \$1.30. What will happen to the optimal solution and profit?

Answer: (*a*) The optimal solution is to manufacture 20,000 jars of applesauce and 59,000 jars of apple juice; and spend\$5,000 to advertise applesauce and \$11,000 to advertise apple juice.

(*b*) The revenue is $20000 \cdot 0.85 + 59000 \cdot 0.90 - 16000 = \$54{,}100$.

(*c*) The advertising budget has a shadow price of \$3.50, meaning that every additional dollar spent on advertising improves profit by \$3.50. Therefore, Eden should definitely spend the additional \$4,000 on advertising. Because we have changed the right-hand side of a constraint, we cannot say what will happen to the solution, and we know that the value of the objective function will increase by $3.50 \cdot 4000 = \$14{,}000$.

(*d*) The advertising minimum on juice advertising has a surplus of 6,000, so removing the minimum will have no affect on juice advertising. If we remove the minimum on applesauce advertising, then the value of the objective function will increase by $1.95 \cdot 5000 = \$9{,}750$. Because we have changed the right-hand side of a constraint—actually, two constraints, but one was not binding—we cannot say what will happen to the optimal solution.

(e) A decrease of the price of a jar of applesauce to \$1.25 is a decrease of \$0.20. This is within the allowable range to decrease the coefficient of applesauce. We therefore know that we will continue to manufacture the same number of jars of applesauce and apple juice, and spend the same amount of money on advertising. Our profit will decrease by $0.20 \cdot 20000 = \$4{,}000$.

(f) We increase the cost of apple juice by \$0.45 decreases the coefficient of apple juice by \$0.45. Because the allowable decrease is \$0.39, we know that the optimal solution will change, but the new solution is unpredictable.

Grade	Number of people
100	1
94	1
92	1
91	1
90	1
88	1
86	1
85	2
78	1
77	2
72	2
70	2
64	1
60	1
57	1
55	1
42	1
35	1

Mean: 74.55
Standard deviation: 16.63