Mathematics 235
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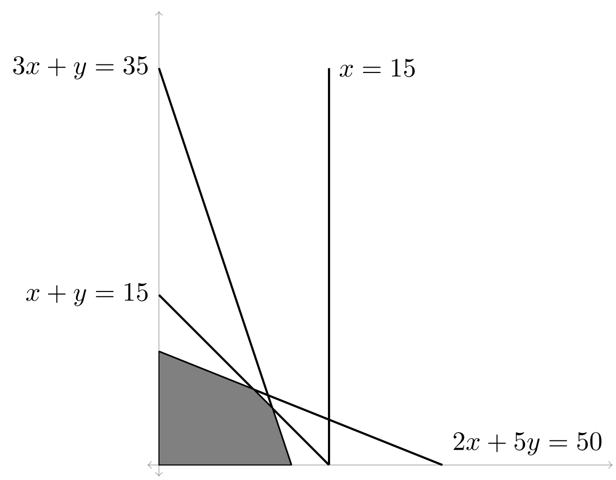
Homework 1
Answers

1. Consider the following linear programming problem:
   Maximize $2x + 3y$ subject to:

$$\begin{array}{rcrcl} x & & & \leq & 15 \\ 2x & + & 5y & \leq & 50 \\ x & + & y & \leq & 15 \\ 3x & + & y & \leq & 35 \\ & & x, y & \geq & 0 \end{array}$$

(a) Use the graphical solution procedure and the extreme points approach, find the optimal solution and the value of the objective function for that optimal solution.
(b) Check your answer to part (a) by using the iso-profit line method of finding the optimal solution.
(c) Determine the amount of slack or surplus for each constraint.
(d) Suppose that the objective function is changed to $\max(5x + 4y)$. Find the optimal solution and value of the objective function for this new problem.
(e) Suppose that the objective function is $2x + cy$. Find a value of the variable c that creates multiple optimal solutions.

Answer: When we graph the inequalities, we see that the feasible region is the shaded one:



The corner points of the region are $(0, 0)$, $(0, 10)$, $(\frac{25}{3}, \frac{20}{3})$, $(10, 5)$, and $(\frac{35}{3}, 0)$. The value of the objective function at each corner point is:

x	y	$2x + 3y$
0	0	0
0	10	30
$\frac{25}{3}$	$\frac{20}{3}$	$\frac{110}{3} \approx 36.67$
10	5	35
$\frac{35}{3}$	0	$\frac{70}{3} \approx 23.33$

The optimal solution is $x = \frac{25}{3} \approx 8.33$ and $y = \frac{20}{3} \approx 6.67$, and the value of the objective function is $\frac{110}{3} \approx 36.67$.

The constraint $x \leq 15$ has $15 - \frac{25}{3} = \frac{20}{3} \approx 6.67$ slack. The constraint $2x + 5y = 50$ is binding. The constraint $x + y = 15$ is binding. The constraint $3x + y \leq 35$ has $\frac{10}{3} \approx 3.33$ slack.

If the objective function is $5x + 4y$, we compute the table again:

x	y	$5x + 4y$
0	0	0
0	10	40
$\frac{25}{3}$	$\frac{20}{3}$	$\frac{205}{3} \approx 68.33$
10	5	70
$\frac{35}{3}$	0	$\frac{175}{3} \approx 58.33$

Now the optimal solution is $x = 10$ and $y = 5$, and the value of this new objective function is 70.

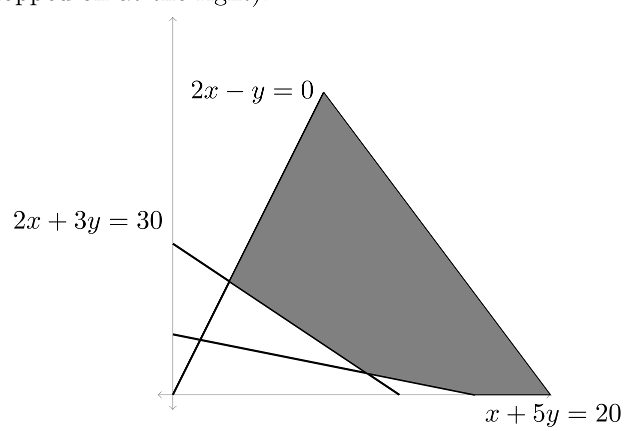
Finally, suppose that the objective function is $2x + cy$. We would like for this function to take the same value at 2 different corner points. One possibility is to use the corner points $(\frac{25}{3}, \frac{20}{3})$ and $(0, 10)$. We want $2 \cdot \frac{25}{3} + \frac{20}{3}c = 10c$. Solving yields $c = 5$. Another possibility is $c = 2$, in which case the objective function has the same values at the corner points $(\frac{25}{3}, \frac{20}{3})$ and $(10, 5)$.

2. Consider the following linear programming problem:
Minimize $4x + 5y$ subject to:

$$\begin{aligned} 2x + 3y &\geq 30 \\ x + 5y &\geq 20 \\ 2x - y &\geq 0 \\ x, y &\geq 0 \end{aligned}$$

(a) Using the graphical solution procedure and *either* the extreme points approach or the iso-profit line, find the optimal solution and the value of the objective function for that optimal solution.
(b) Determine the amount of slack or surplus for each constraint.

Answer: When we graph the inequalities, we see that the feasible region is the shaded one (except that it is infinitely large, and not chopped off at the right):



The corner points are $(\frac{15}{4}, \frac{15}{2})$, $(\frac{90}{7}, \frac{10}{7})$, and $(20, 0)$. We compute

x	y	$4x + 5y$
$\frac{15}{4}$	$\frac{15}{2}$	$\frac{105}{2} = 52.5$
$\frac{90}{7}$	$\frac{10}{7}$	$\frac{410}{7} \approx 58.57$
20	0	80

We see that the minimum value occurs when $x = \frac{15}{4} = 3.75$ and $y = \frac{15}{2} = 7.5$, and the value of the objective function is 52.5.

The constraint $2x + 3y \geq 30$ is binding. The constraint $x + 5y \geq 20$ has a surplus of $\frac{85}{4} = 21.25$. The constraint $2x - y \geq 0$ is binding.

3. The **Oak Works** manufactures handcrafted chairs and tables from locally and sustainably grown oak. Each month, they have 2,500 pounds of oak available. Each chair uses 25 pounds of oak, and each table uses 50 pounds. Each table and each chair requires 6 hours of labor, and there are 480 hours of labor available each month. Each table has a profit of \$400, and each chair has a profit of \$100. The company wants to produce *at least* twice as many chairs as tables.

Assume that the company is interested in maximizing the total profit contribution.

(a) Formulate a mathematical model for this problem.
(b) Find the optimal solution using the graphical solution procedure and *either* the extreme points approach or the iso-profit line. How many chairs and how many tables should be manufactured? You may assume that manufacture of fractional quantities is permissible.
(c) What is the total profit contribution with the above production quantities?
(d) Find other values for the profit of the two products that would make the optimal solution producing only chairs.

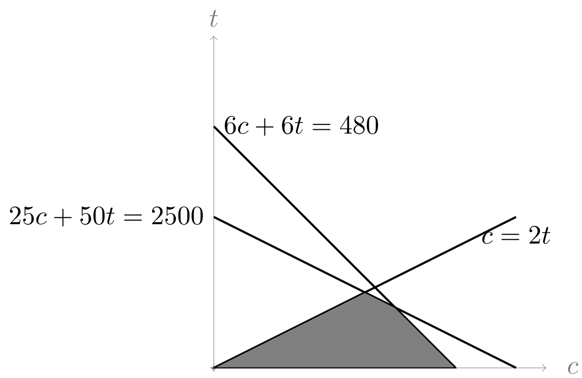
Answer: Let

$$c = \text{the number of chairs manufactured}$$
$$t = \text{the number of tables manufactured}$$

We need to maximize $100c + 400t$, given

Pounds of oak:	$25c + 50t \leq 2500$
Labor:	$6c + 6t \leq 480$
At least twice as many chairs:	$c \geq 2t$

The optimal region is shaded:



The corner points are $(0, 0)$, $(50, 25)$, $(60, 20)$, and $(80, 0)$. We have

c	t	$100c + 400t$
0	0	0
50	25	15000
60	20	14000
80	0	8000

We see that the maximum profit occurs when we manufacture 50 chairs and 25 tables, for a total profit of \$15000.

If the profits are reversed, and we make \$400 for every chair and \$100 for every table, then the table becomes

c	t	$400c + 100t$
0	0	0
50	25	22500
60	20	26000
80	0	32000

and now the optimal solution is to make 80 chairs and no tables.

4. **Ralph** loves steak and potatoes so much that he has decided to consume only those two foods for the rest of his life. (Ralph is a stubborn teen-ager.) He is aware that this is not the healthiest possible choice of diet, so he wants to satisfy least some key nutritional requirements. He has obtained this nutrition and cost information:

Grams of Ingredient per Serving			
Ingredient	**Steak**	**Potatoes**	**Daily Requirement (grams)**
Carbohydrate	5	15	≥ 50
Protein	20	5	≥ 40
Fat	15	2	≤ 60
Cost per serving	\$4	\$2	

Ralph wishes to find the number of servings of each food (possibly fractional) to satisfy these three dietary requirements at minimal cost.

Formulate and graphically solve a mathematical model for this problem.

Answer: Let

$$s = \text{the number of servings of steak}$$
$$p = \text{the number of servings of potatoes}$$

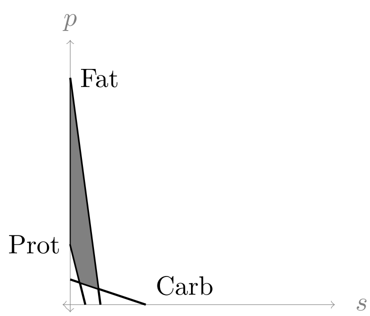
We need to minimize $4s + 2p$ given

Carbohydrates: $5s + 15p \geq 50$

Protein: $20s + 5p \geq 40$

Fat: $15s + 2p \leq 60$

The optimal region is shaded:



The corner points are $(\frac{14}{11}, \frac{32}{11}) \approx (1.27, 2.91)$, $(0, 8)$, $(0, 30)$, and $(\frac{160}{43}, \frac{90}{43}) \approx (3.72, 2.09)$. The values of the objective function are

s	p	$4s + 2p$
$\frac{14}{11}$	$\frac{32}{11}$	$\frac{120}{11} \approx 10.91$
0	8	16
0	30	60
$\frac{160}{43}$	$\frac{90}{43}$	$\frac{820}{43} \approx 19.07$

The minimal value is therefore to eat $\frac{14}{11}$ servings of steak and $\frac{32}{11}$ servings of potatoes for a cost of approximately \$10.91.