Mathematics 235
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Homework 3
Answers

1. For the past two weeks, we have studied the following linear programming problem: Maximize $2x + 3y$ subject to:

$$\begin{array}{rcrcr} x & & & \leq & 15 \\ 2x & + & 5y & \leq & 50 \\ x & + & y & \leq & 15 \\ 3x & + & y & \leq & 35 \\ & & x, y & \geq & 0 \end{array}$$

(a) Draw the graph of the feasible region that you produced as part of the first homework assignment. (Either use graph paper, or else work carefully and neatly on ordinary paper.) Label the coordinates of all of the corners (vertices) of the feasible region. Label the optimal solution. You can find all of this information in earlier homework answers, so there is no credit for this part of the problem.

(b) Suppose that the objective function changes to $2x + Cy$. Fill in the blanks in the following sentence, and explain your reasoning by using graphical/algebraic sensitivity analysis:

The optimal solution will not change if C varies from ________ to ________.

(c) Suppose that the objective function changes to $Dx + 3y$. Fill in the blanks in the following sentence, and explain your reasoning by using graphical/algebraic sensitivity analysis:

The optimal solution will not change if D varies from ________ to ________.

(d) What is the shadow price associated with the constraint

$$x \leq 15?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

(e) What is the shadow price associated with the constraint

$$2x + 5y \leq 50?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

(f) What is the shadow price associated with the constraint

$$x + y \leq 15?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

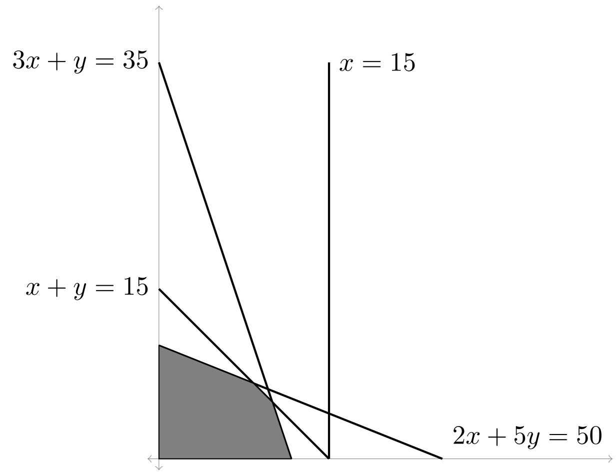
(g) What is the shadow price associated with the constraint

$$3x + y \leq 35?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

(h) Use the *Excel* spreadsheet that you used to answer Homework 2 and produce a sensitivity report using *Solver*. Print and attach a copy of the sensitivity report, and explain which numbers in the sensitivity report correspond to the computations that you did above.

Answer: (a) The feasible region is shaded:



The corner points of the region are $(0, 0)$, $(0, 10)$, $(\frac{25}{3}, \frac{20}{3})$, $(10, 5)$, and $(\frac{35}{3}, 0)$. The optimal solution is $x = \frac{25}{3}$ and $y = \frac{20}{3}$.

(b) The two binding constraints are $2x + 5y = 50$ and $x + y = 15$. The first has slope $-\frac{2}{5}$ and the second has slope -1. If the objective function is $2x + Cy$, the slope of an isoprofit line is $-\frac{2}{C}$. So we have

$$-\frac{2}{5} \geq -\frac{2}{C} \geq -1$$
$$\frac{2}{5} \leq \frac{2}{C} \leq 1$$
$$\frac{5}{2} \geq \frac{C}{2} \geq 1$$
$$5 \geq C \geq 2$$

Therefore, C can be as small as 2 and as large as 5 without changing the optimal solution.

(c) Now the slope of an isoprofit line for the objective function $Dx + 3y$ is $-\frac{D}{3}$, so we have

$$-\frac{2}{5} \geq -\frac{D}{3} \geq -1$$
$$\frac{2}{5} \leq \frac{D}{3} \leq 1$$
$$\frac{6}{5} \leq D \leq 3$$

Therefore, D can be as small as $\frac{6}{5}$ or as large as 3 and the optimal solution will not change.

(d) The constraint $x \leq 15$ is not binding, so the shadow price is 0. The right-hand side of the constraint can decrease to $\frac{25}{3}$, a decrease of $\frac{20}{3}$, before the shadow price changes. The right-hand side of the constraint can increase infinitely.

(e) The constraint $2x + 5y \leq 50$ is binding. We simultaneously solve $2x + 5y = 50 + \lambda$ and $x + y = 15$. Multiply the second equation by 2 to get $2x+2y = 30$, and subtract to get $3y = 20+\lambda$, so $y = \frac{20}{3}+\frac{1}{3}\lambda$. We have $x = 15-y = 15-(\frac{20}{3}+\frac{1}{3}\lambda) = \frac{25}{3}-\frac{1}{3}\lambda$. The objective function is $2x+3y = 2(\frac{25}{3}-\frac{1}{3}\lambda)+3(\frac{20}{3}+\frac{1}{3}\lambda) = \frac{110}{3}+\frac{1}{3}\lambda$. Therefore, the shadow price of the constraint is $\frac{1}{3}$.

To find the feasible range, we first have to use $x \geq 0$, implying that $\frac{25}{3} - \frac{1}{3}\lambda \geq 0$, or $\frac{25}{3} \geq \frac{1}{3}\lambda$, or $25 \geq \lambda$. In the other direction, we use $3x + y \leq 35$, or $3(\frac{25}{3} - \frac{1}{3}\lambda) + (\frac{20}{3} + \frac{1}{3}\lambda) \leq 35$. This simplifies to $\frac{95}{3} - \frac{2}{3}\lambda \leq 35$, or $-\frac{2}{3}\lambda \leq \frac{10}{3}$, or $\lambda \geq -5$.

(f) We write $x + y = 15 + \lambda$ and solve simultaneously with $2x + 5y = 50$. Multiply the first equation by 2 to get $2x + 2y = 30 + 2\lambda$, and subtraction yields $3y = 20 - 2\lambda$, or $y = \frac{20}{3} - \frac{2}{3}\lambda$. We have $x = 15 + \lambda - y = 15 + \lambda - (\frac{20}{3} - \frac{2}{3}\lambda) = \frac{25}{3} + \frac{5}{3}\lambda$. We now have $2x + 3y = 2(\frac{25}{3} + \frac{5}{3}\lambda) + 3(\frac{20}{3} - \frac{2}{3}\lambda) = \frac{110}{3} + \frac{4}{3}\lambda$. Therefore, the shadow price of the constraint is $\frac{4}{3}$.

To compute the feasible range, we first use $x \geq 0$ to get $\frac{25}{3} + \frac{5}{3}\lambda \geq 0$, or $\frac{5}{3}\lambda \geq -\frac{25}{3}$, and so $\lambda \geq -5$. In the other direction, we use $3x + y \leq 35$, and get $3(\frac{25}{3} + \frac{5}{3}\lambda) + (\frac{20}{3} - \frac{2}{3}\lambda) \leq 35$. This simplifies to $\frac{95}{3} + \frac{13}{3}\lambda \leq 35$, or $\frac{13}{3}\lambda \leq \frac{10}{3}$, or $\lambda \leq \frac{10}{13} \approx 0.7692$.

(g) The constraint $3x + y \leq 35$ is slack, so the shadow price is 0. The slack is $\frac{10}{3}$, so the right-hand side of the constraint can decrease by $\frac{10}{3}$ and increase infinitely.

2. For the past two weeks, we have studied the following linear programming problem: Minimize $4x + 5y$ subject to:

$$\begin{array}{rcrcr} 2x & + & 3y & \geq & 30 \\ x & + & 5y & \geq & 20 \\ 2x & - & y & \geq & 0 \\ & & x, y & \geq & 0 \end{array}$$

(a) Draw the graph of the feasible region that you produced as part of the first homework assignment. (Either use graph paper, or else work carefully and neatly on ordinary paper.) Label the coordinates of all of the corners (vertices) of the feasible region. Label the optimal solution. You can find all of this information in earlier homework answers, so there is no credit for this part of the problem.

(b) Suppose that the objective function changes to $4x + Cy$. Fill in the blanks in the following sentence, and explain your reasoning by using graphical/algebraic sensitivity analysis:

The optimal solution will not change if C varies from ________ to ________.

(c) Suppose that the objective function changes to $Dx + 5y$. Fill in the blanks in the following sentence, and explain your reasoning by using graphical/algebraic sensitivity analysis:

The optimal solution will not change if D varies from ________ to ________.

(d) What is the shadow price associated with the constraint

$$2x + 3y \geq 30?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

(e) What is the shadow price associated with the constraint

$$x + 5y \geq 20?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

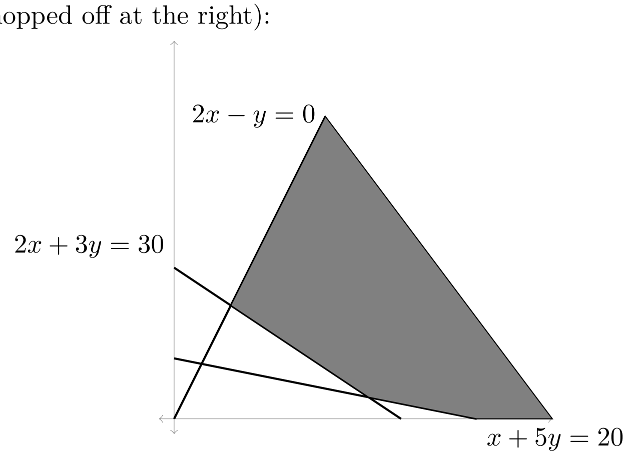
(f) What is the shadow price associated with the constraint

$$2x - y \geq 0?$$

What is the range of feasibility of this shadow price? Compute these answers using algebra and graphical methods.

(g) Use the *Excel* spreadsheet that you used to answer Homework 2 and produce a sensitivity report using *Solver*. Print and attach a copy of the sensitivity report, and explain which numbers in the sensitivity report correspond to the computations that you did above.

Answer: When we graph the inequalities, we see that the feasible region is the shaded one (except that it is infinitely large, and not chopped off at the right):



(a) The corner points are $(\frac{15}{4}, \frac{15}{2})$, $(\frac{90}{7}, \frac{10}{7})$, and $(20, 0)$. The minimum value occurs when $x = \frac{15}{4} = 3.75$ and $y = \frac{15}{2} = 7.5$, and the value of the objective function is 52.5. The two binding constraints are $2x+3y = 30$ and $2x - y = 0$, with slope $-\frac{2}{3}$ and 2.

(b) An isoprofit line for the objective function $4x + Cy$ has slope $-\frac{4}{C}$, so we need to have

$$\begin{aligned} -\frac{4}{C} &\leq -\frac{2}{3} \\ \frac{4}{C} &\geq \frac{2}{3} \\ \frac{C}{4} &\leq \frac{3}{2} \\ C &\leq 6 \end{aligned}$$

In our terms, we insist that $C \geq 0$, to provide a lower bound. Alternatively, we can solve $-\frac{4}{C} \geq 2$, or $\frac{C}{4} \geq -\frac{1}{2}$, or $C \geq -2$. This is the answer in *Excel*'s sensitivity report.

(c) An isoprofit line for the objective function $Dx + 5y$ has slope $-\frac{D}{5}$, so we solve

$$\begin{aligned} -\frac{D}{5} &\leq -\frac{2}{3} \\ \frac{D}{5} &\geq \frac{2}{3} \\ D &\geq \frac{10}{3} \end{aligned}$$

There is no upper bound.

(d) The constraint $2x + 3y \leq 30$ is binding, so we solve $2x + 3y = 30 + \lambda$ simultaneously with $2x - y = 0$. Substitute $y = 2x$ into the first equation, and we have $8x = 30 + \lambda$, or $x = \frac{15}{4} + \frac{1}{8}\lambda$. We also have $y = 2x = \frac{15}{2} + \frac{1}{4}\lambda$. Substitute into the objective function, and we get $4x + 5y = 4(\frac{15}{4} + \frac{1}{8}\lambda) + 5(\frac{15}{2} + \frac{1}{4}\lambda) = \frac{105}{2} + \frac{7}{4}\lambda$, so the shadow price is $\frac{7}{4}$.

To compute the feasible range, we use $x + 5y \geq 20$, and get $\frac{15}{4} + \frac{1}{8}\lambda + 5(\frac{15}{2} + \frac{1}{4}\lambda) \geq 20$, or $\frac{165}{4} + \frac{11}{8}\lambda \geq 20$, or $\frac{11}{8}\lambda \geq -\frac{85}{4}$, or $\lambda \geq -\frac{170}{11} \approx -15.4545$. There is no upper bound.

(e) The constraint $x + 5y \geq 20$ is not binding. It has a shadow price of 0. The surplus is $21\frac{1}{4}$, so the right-hand side can increase by $21\frac{1}{4}$ and decrease arbitrarily (or to 0) without changing the shadow price.

(f) We solve $2x - y = \lambda$ simultaneously with $2x + 3y = 30$. We substitute $y = 2x - \lambda$ and get $2x + 3(2x - \lambda) = 30$, or $8x - 3\lambda = 30$, or $x = \frac{15}{4} + \frac{3}{8}\lambda$. We have $y = 2x - \lambda = 2(\frac{15}{4} + \frac{3}{8}\lambda) - \lambda = \frac{15}{2} - \frac{1}{4}\lambda$. Therefore, $4x + 5y = 4(\frac{15}{4} + \frac{3}{8}\lambda) + 5(\frac{15}{2} - \frac{1}{4}\lambda) = \frac{105}{2} + \frac{1}{4}\lambda$. Therefore, the shadow price is $\frac{1}{4}$.

To find the feasible range, we first use $x \geq 0$ to get $\frac{15}{4} + \frac{3}{8}\lambda \geq 0$, or $\frac{3}{8}\lambda \geq -\frac{15}{4}$, or $\lambda \geq -10$. In the other direction, we need to use $x + 5y \geq 20$: $(\frac{15}{4} + \frac{3}{8}\lambda) + 5(\frac{15}{2} - \frac{1}{4}\lambda) \geq 20$, which simplifies to $\frac{165}{4} - \frac{7}{8}\lambda \geq 20$, or $-\frac{7}{8}\lambda \geq -\frac{85}{4}$, or $\lambda \leq \frac{170}{7} \approx 24.2857$.

3. Last week, we solved the following problem using *Excel*:

> **Ralph** loves steak and potatoes so much that he has decided to consume only those two foods for the rest of his life. (Ralph is a stubborn teen-ager.) He is aware that this is not the healthiest possible choice of diet, so he wants to satisfy least some key nutritional requirements. He has obtained this nutrition and cost information:

Grams of Ingredient per Serving			
Ingredient	**Steak**	**Potatoes**	**Daily Requirement (grams)**
Carbohydrate	5	15	≥ 50
Protein	20	5	≥ 40
Fat	15	2	≤ 60
Cost per serving	\$4	\$2	

> Ralph wishes to find the number of servings of each food (possibly fractional) to satisfy these three dietary requirements at minimal cost.

Use your spreadsheet to generate a sensitivity report, and use the sensitivity report to answer the following questions. Please note that perhaps not all of the questions can be answered by using the sensitivity report; if that is the case, say so. Print out the sensitivity report and attach it to your answers.

Please note that each of these questions suggests an independent change to the problem.

(a) What happens to the optimal solution and daily cost if the daily maximum of fat is decreased to 50 grams?

(b) What happens to the optimal solution and daily cost if the cost of a serving of steak increases to \$10?

(c) What happens to the optimal solution and daily cost if the daily minimum of protein is decreased to 30 grams?

(d) What happens to the optimal solution and daily cost if the daily minimum of protein is decreased to 10 grams?

(e) What happens to the optimal solution and daily cost if the cost of a serving of potatoes increases to \$3?

Answer: (a) The shadow price of the fat constraint is 0, with an allowable decrease of 35.09, which is larger than 10. Therefore, neither the optimal solution nor the cost change if the fat maximum decreases to 50.

(b) The current cost of a serving of steak is \$4, and the allowable increase on that coefficient is 4. Therefore, we can say absolutely nothing about what happens to the optimal solution if the cost of a serving of steak goes up to \$10, other than that it will change.

(c) The shadow price of the daily protein minimum is 0.1818. If the daily minimum decreases by 10, then the cost will decrease by $10 \cdot 0.1818 = \$18.18$. We cannot say anything about what happens to the optimal solution.

(d) If the daily protein minimum decreases to 10 grams, we cannot say what will happen, because that is a decrease of 30, and the allowable decrease is 23.33. We can be sure that the solution will change, but cannot say more.

(e) The allowable increase on the cost of potatoes is \$10, so an increase of \$1 is within the optimal range. The optimal solution will not change, and the cost will increase by \$2.90909.

4. **Oak Works** has an enormous backlog of orders for hand-made chairs and tables. The company has decided to hire three carpenters as subcontractors to take care of these orders. The three carpenters each offers an estimate how many hours it will take to build the necessary chairs and tables, along with how many hours each actually has available during the next week, and the hourly cost:

	Carpenter A	**Carpenter B**	**Carpenter C**
Hours to complete all tables	50	42	30
Hours to complete all chairs	60	48	35
Hours available	40	30	35
Hourly wage	\$36	\$42	\$55

The tables tells us that the Carpenter B estimates that it would take him 42 hours to make all of the necessary tables. However, because he only has 30 hours available next week, if he only builds tables, he can only manufacture $\frac{30}{42}$ of the necessary number of orders.

(a) Formulate a linear programming model to determine what fraction of the tables and what fraction of the chairs should be built by each of the 3 carpenters so that the entire backlog can be cleared up at minimal cost.

(b) Solve your linear program using *Excel* and *Solver*. State your solution, including the final cost, in whole sentences.

Answer the next three questions using the sensitivity report produced by *Solver*. They pose three independent scenarios.

(c) Suppose that Carpenter A had 35 hours available. Say as much possible about how the solution and total cost will change.

(d) Suppose that Carpenter B had 35 hours available. Say as much possible about how the solution and total cost will change.

(e) Suppose that Carpenter C lowers his hourly rate to \$54. Say as much possible about how the solution and total cost will change.

Answer: (a) Let

$$\begin{aligned}
c_A &= \text{the number of hours Carpenter A works on chairs}\\
t_A &= \text{the number of hours Carpenter A works on tables}\\
c_B &= \text{the number of hours Carpenter B works on chairs}\\
t_B &= \text{the number of hours Carpenter B works on tables}\\
c_C &= \text{the number of hours Carpenter C works on chairs}\\
t_C &= \text{the number of hours Carpenter C works on tables}
\end{aligned}$$

We need to minimize $36(c_A + t_A) + 42(c_B + t_B) + 55(c_C + t_C)$ given

Carpenter A hours:	$c_A + t_A \leq 40$
Carpenter B hours:	$c_B + t_B \leq 42$
Carpenter C hours:	$c_C + t_C \leq 40$
Finish all tables:	$\frac{1}{50}t_A + \frac{1}{42}t_B + \frac{1}{30}t_C = 1$
Finish all chairs:	$\frac{1}{60}c_A + \frac{1}{48}c_B + \frac{1}{35}c_C = 1$

(b) *Excel* reports that the optimal solution is that Carpenter A works on tables for 13.54167 hours, Carpenter B works on chairs for 30 hours, and Carpenter C works on tables for 21.875 hours and on chairs for 13.125 hours. In other words, Carpenter A makes $\frac{13.54167}{50} \approx 0.2708$ of the tables, Carpenter B makes $\frac{30}{48} = 0.6250$ of the chairs, and Carpenter C makes $\frac{21.875}{30} \approx 0.7292$ of the tables and $\frac{13.125}{35} = 0.3750$ of the chairs.

(c) The shadow price for the constraint on carpenter A's hours is 0. He is only using 13.54167 hours, so decreasing his hours to 35 changes neither the total cost nor the optimal solution.

(d) If Carpenter B has 35 hours available, the optimal solution will change, but we cannot say how. The shadow price on this constraint is -1.75, so the cost will change by $5(-1.75) = \$8.75$.

(e) Surprisingly, this question cannot be answered, at least if you use the formulation that I gave above. The carpenter's hourly rate is the coefficient of two different decision variables, so changing the rate changes two different coefficients simultaneously. It is impossible to say what will happen.

There are alternative ways to formulate the problem that do allow you to answer the question, because in those ways the carpenter's hourly rate is a coefficient of only one variable in the objective function.