Mathematics 235
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Homework 6
Answers

1. **Pearson Annuities** has the following financial obligations to meet over the next four years, in thousands of dollars:

Year	1	2	3	4
Payment	110	125	500	100

Each payment is due at the start of the year. Pearson can purchase at the start of year 1 any or all of the following three bonds to help meet these obligations. Each bond has a par value of $1,000.

Security	**Current Price**	**Rate (%)**	**Years to Maturity**
A	$1020	1.55	1
B	$1025	3.85	2
C	$1015	3.55	3

Pearson also has available a savings account, with a guaranteed rate of 1% annually.

(a) Formulate a linear program that minimizes the cost of meeting these obligations.

(b) Use *Excel* and *Solver* to find the optimal solution. How many of each bond should be purchased, and how much money should be set aside in savings?

(c) Now impose the constraint that bonds cannot be purchased in fractional amounts, and resolve the problem using *Excel* and *Solver*. What is the new solution?

(d) Finally, impose a constraint that at most 475 of each of the three bonds can be purchased, keeping the integer constraint as well. Resolve the problem using *Excel* and *Solver*. What is the new solution?

Answer: We work in thousands of dollars. Let

$$\begin{aligned}
A &= \text{number of securities of type A purchased}\\
B &= \text{number of securities of type B purchased}\\
C &= \text{number of securities of type C purchased}\\
S_i &= \text{amount of money in savings at end of year } i,\ i = 0, 1, 2, 3.
\end{aligned}$$

We wish to minimize S_0, given

$$\begin{aligned}
&\text{Year 1:} & S_0 - 1.02A - 1.025B - 1.015C - 110 &= S_1\\
&\text{Year 2:} & 1.01S_1 + .0155A + .0385B + .0355C + A - 125 &= S_2\\
&\text{Year 3:} & 1.01S_2 + .0385B + .0355C + B - 500 &= S_3\\
&\text{Year 4:} & 1.01S_3 + .0355C + C - 100 &\geq 0
\end{aligned}$$

Excel reports that the optimal solution is to start with approximately $800,277.8312 in savings, and purchase no bonds of type A, approximately 478.162 bonds of type B, and approximately 96.572 bonds of type C.

If we impose the restriction that bonds must be purchased in integer amounts—i.e., the variables A, B, and C are restricted to be integers—then *Excel* reports that the optimal solution is to start with approximately $800,317.1921 in savings, and purchase purchase no bonds of type A, 478 bonds of type B, and 96 bonds of type C.

Finally, if we impose the constraints that $A \leq 475$, $B \leq 475$, and $C \leq 475$, *Excel* reports that the solution is now to start with $800,410.6608 in savings, and purchase no type A bonds, 475 type B bonds, and 96 type C bonds.

2. **White Dairies** has farms in Attleboro, Boxborough, Chelmsford, and Dartmouth, and must ship milk to bottling plants in Gloucester, Haverhill, and Ipswich. The cost of shipping is \$0.01/gallon/mile, and the distances from each farm to each plant are:

		Plants	
Farms	Gloucester	Haverhill	Ipswich
Attleboro	73.8	73.6	67.7
Boxborough	60.9	36.2	47.6
Chelmsford	47.4	22.7	34.2
Dartmouth	95.1	94.9	88.9

Each farm supplies 6,000 gallons daily, and each bottling plant can bottle up to 9,000 gallons daily. All milk must be bottled.

(a) Formulate a linear model to bottle all of the milk at minimal cost.
(b) Solve your linear model using *Excel* and *Solver*.
(c) Suppose that White could expand one of the three bottling plants to have a capacity of 10,000 gallons daily. Assuming that the cost of the expansion is the same at each of the three plants, which (if any) of the plants should be expanded? Your answer should be based on the sensitivity report from your *Solver* solution.

Answer: Let

$$x_{ij} = \text{gallons shipped from location } i \text{ to location } j,\ i = A, B, C, D,\ j = G, H, I$$

We wish to minimize

$$0.01(73.8x_{AG} + 73.6x_{AH} + 67.7x_{AI} + 60.9x_{BG} + 36.2x_{BH} + 47.6x_{BI}+ \\ 47.4x_{CG} + 22.7x_{CH} + 34.2x_{CI} + 95.1x_{DG} + 94.9x_{DH} + 88.9x_{DI})$$

given

Attleboro:	$x_{AG} + x_{AH} + x_{AI} = 6000$
Boxborough:	$x_{BG} + x_{BH} + x_{BI} = 6000$
Chelmsford:	$x_{CG} + x_{CH} + x_{CI} = 6000$
Dartmouth:	$x_{DG} + x_{DH} + x_{DI} = 6000$
Gloucester:	$x_{AG} + x_{BG} + x_{CG} + x_{DG} \leq 9000$
Haverhill:	$x_{AH} + x_{BH} + x_{CH} + x_{DH} \leq 9000$
Ipswich:	$x_{AI} + x_{BI} + x_{CI} + x_{DI} \leq 9000$

Excel reports that the optimal solution is to ship 6,000 gallons from Attleboro to Gloucester, 3,000 gallons from Boxborough to Haverhill, 3,000 gallons from Boxborough to Ipswich, 6,000 gallons from Chelmsford to Haverhill, and 6,000 gallons from Dartmouth to Ipswich, for a total cost of \$13,638.

The shadow price associated to the Haverhill plant is -0.175, and the shadow price associated to the Ipswich plant is -0.061. Therefore, expanding the Haverhill plant would save more money, and the allowable increase on that shadow price is 3000, so we know that our analysis is valid.

3. The **FDA** issues a ruling that all milk must be pasteurized at central locations before being bottled, thereby complicating White Dairy's planning. Pasteurization plants are located in Essex and Fitchburg. The relevant distances are in the table. The production at each farm remains 6,000 gallons daily, and that the capacity of the bottling plants remains 9,000 gallons.

(a) Formulate a linear model to pasteurize and bottle all of the milk at minimal cost.
(b) Solve your linear model using *Excel* and *Solver*.
(c) We have not yet imposed any limitation on the processing capacity of the pasteurization plants. Suppose that each plant can pasteurize a maximum of 15,000 gallons/day. Add this

Table: **Distances from farms to bottling plants**

	Pasteurization	
Farms	Essex	Fitchburg
Attleboro	71.0	60.7
Boxborough	58.1	19.8
Chelmsford	44.6	28.1
Dartmouth	92.3	95.9

	Bottling		
Pasteurization	Gloucester	Haverhill	Ipswich
Essex	7.4	21.3	5.6
Fitchburg	74.6	49.9	61.4

constraint to your mathematical formulation, and solve the modified problem using *Excel* and *Solver*.

Answer: Again, we set x_{ij} to be the number of gallons shipped from location i to location j. We wish to minimize

$$0.01(71.0x_{AE} + 60.7x_{AF} + 58.1x_{BE} + 19.8x_{BF} + 44.6x_{CE} + 28.1x_{CF} + 92.3x_{DE} + 95.9x_{DF} + 7.4x_{EG} + 21.3x_{EH} + 5.6x_{EI} + 74.6x_{FG} + 49.9x_{FH} + 61.4x_{FI})$$

The constraints are:

Attleboro:	$x_{AE} + x_{AF} = 6000$
Boxborough:	$x_{BE} + x_{BF} = 6000$
Chelmsford:	$x_{CE} + x_{CF} = 6000$
Dartmouth:	$x_{DE} + x_{DF} = 6000$
Essex:	$x_{AE} + x_{BE} + x_{CE} + x_{DE} = x_{EG} + x_{EH} + x_{EI}$
Fitchburg:	$x_{AF} + x_{BF} + x_{CF} + x_{DF} = x_{FG} + x_{FH} + x_{FI}$
Gloucester:	$x_{EG} + x_{FG} \leq 9000$
Haverhill:	$x_{EH} + x_{FH} \leq 9000$
Ipswich:	$x_{EI} + x_{FI} \leq 9000$

Excel reports that the optimal solution is to ship 6000 gallons from Attleboro to Essex, 6000 gallons from Boxborough to Fitchburg, 6000 gallons from Chelmsford to Essex, 6000 gallons from Dartmouth to Essex, 9000 gallons from Essex to Gloucester, 9000 gallons from Essex to Ipswich, and 6000 gallons from Fitchburg to Haverhill, for a total cost of $17,826.

If the plants have a capacity of 15000 gallons, we must impose additional constraints:

Essex:	$x_{AE} + x_{BE} + x_{CE} + x_{DE} \leq 15000$
Fitchburg:	$x_{AF} + x_{BF} + x_{CF} + x_{DF} \leq 15000$

Excel reports that the new solution is to ship 6000 gallons from Attleboro to Essex, 6000 gallons from Boxborough to Fitchburg, 3000 gallons from Chelmsford to Essex, 3000 gallons from Chelmsford to Fitchburg, 6000 gallons from Dartmouth to Essex, 6000 gallons from Essex to Gloucester, 9000 gallons from Essex to Ipswich, and 9000 gallons from Fitchburg to Haverhill, for a total cost of $18,606.

4. **Barton's Groceries** is contracting with different vendors to replace the roof at each of their four stores. Barton's has received six bids, and wants to try a different bidder at each of their four locations. The bids are:

		Location		
Bidder	1	2	3	4
A	190	175	125	230
B	150	235	155	220
C	210	225	135	260
D	170	185	190	280
E	220	190	140	240
F	270	200	130	260

(a) Formulate a linear model to help Barton's choose the four roofers while minimizing total cost

(b) Solve your model using *Excel* and *Solver*.

(c) Suppose that Barton's relaxes the requirement, and asks instead that at least two different roofers must be used for the four jobs. A bidder is now allowed to replace the roof at two or even three of the four stores. Reformulate the problem, and solve the modified problem using *Excel* and *Solver*.

Answer: (a) Let

$$y_{ij} = \begin{cases} 1 & \text{if bidder } i \text{ is accepted for location } j \\ 0 & \text{if bidder } i \text{ is } \textit{not} \text{ accepted for location } j \end{cases}$$

where $i = A, B, C, D, E,$ or F, and $j = 1, 2, 3,$ or 4. We need to minimize

$$190y_{A1} + y_{A2}175 + 125y_{A3} + 230y_{A4} + 150y_{B1} + 235y_{B2} + 155y_{B3} + 220y_{B4}$$
$$+$$
$$210y_{C1} + 225y_{C2} + 135y_{C3} + 260y_{C4} + 170y_{D1} + 185y_{D2} + 190y_{D3} + 280y_{D4}$$
$$+$$
$$220y_{E1} + 190y_{E2} + 140y_{E3} + 240y_{E4} + 270y_{F1} + 200y_{F2} + 130y_{F3} + 260y_{F4}$$

given

Location 1:	$y_{A1} + y_{B1} + y_{C1} + y_{D1} + y_{E1} + y_{F1} = 1$
Location 2:	$y_{A2} + y_{B2} + y_{C2} + y_{D2} + y_{E2} + y_{F2} = 1$
Location 3:	$y_{A3} + y_{B3} + y_{C3} + y_{D3} + y_{E3} + y_{F3} = 1$
Location 4:	$y_{A4} + y_{B4} + y_{C4} + y_{D4} + y_{E4} + y_{F4} = 1$
Bidder A:	$y_{A1} + y_{A2} + y_{A3} + y_{A4} \leq 1$
Bidder B:	$y_{B1} + y_{B2} + y_{B3} + y_{B4} \leq 1$
Bidder C:	$y_{C1} + y_{C2} + y_{C3} + y_{C4} \leq 1$
Bidder D:	$y_{D1} + y_{D2} + y_{D3} + y_{D4} \leq 1$
Bidder E:	$y_{E1} + y_{E2} + y_{E3} + y_{E4} \leq 1$
Bidder F:	$y_{F1} + y_{F2} + y_{F3} + y_{F4} \leq 1$

Because this is a transportation problem, it actually is unnecessary to add the constraints that y_{ij} are binary variables, though it is not wrong to do so.

(b) *Excel* and *Solver* report that the optimal solution is to assign Location 1 to bidder B, location 2 to bidder A, location 3 to bidder F, and location 4 to bidder E, for a total cost of \$695. The sensitivity report shows that there are multiple optimal solutions, incidentally.

(c) We keep the variables and objective function the same as above, but alter the final 6 constraints to be:

Bidder A:	$y_{A1} + y_{A2} + Y_{A3} + y_{A4} \leq 3$
Bidder B:	$y_{B1} + y_{B2} + Y_{B3} + y_{B4} \leq 3$
Bidder C:	$y_{C1} + y_{C2} + Y_{C3} + y_{C4} \leq 3$
Bidder D:	$y_{D1} + y_{D2} + Y_{D3} + y_{D4} \leq 3$
Bidder E:	$y_{E1} + y_{E2} + Y_{E3} + y_{E4} \leq 3$
Bidder F:	$y_{F1} + y_{F2} + Y_{F3} + y_{F4} \leq 3$

Excel and *Solver* report that the optimal solution now is to assign locations 1 and 4 to bidder B, and locations 2 and 3 to bidder A, for a total cost of \$670. This time the sensitivity report claims that this is the only solution.