Mathematics 235
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Homework 7
Answers

1. **White Dairies** has farms in Attleboro, Boxborough, Chelmsford, and Dartmouth, and must ship milk to bottling plants in Gloucester, Haverhill, and Ipswich. The cost of shipping is \$0.01/gallon/mile, and the distances from each farm to each plant are:

TABLE: **Distances from farms to bottling plants**

		Plants	
Farms	Gloucester	Haverhill	Ipswich
Attleboro	73.8	73.6	67.7
Boxborough	60.9	36.2	47.6
Chelmsford	47.4	22.7	34.2
Dartmouth	95.1	94.9	88.9

Each farm supplies 6,000 gallons daily, and each bottling plant can bottle up to 9,000 gallons daily. All milk must be bottled.

(*a*) Suppose that if a route is used, the minimum shipment on that route is 2,000 gallons. (This does not mean that every route must be used, only that if a route is used at all, it must be used for a minimum of 2,000 gallons.) In addition, the maximum shipment on any route is 5,000 gallons. Formulate a linear program to ship the milk at minimal cost.

(*b*) Now use *Excel* and *Solver* to find the optimal solution.

Answer: Let

$$x_{ij} = \text{gallons shipped from location } i \text{ to location } j,\ i = A, B, C, D,\ j = G, H, I$$

$$y_{ij} = \begin{cases} 1 & \text{if } x_{ij} \neq 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases}$$

We wish to minimize

$$0.01(73.8x_{AG} + 73.6x_{AH} + 67.7x_{AI} + 60.9x_{BG} + 36.2x_{BH} + 47.6x_{BI} + 47.4x_{CG} + 22.7x_{CH} + 34.2x_{CI} + 95.1x_{DG} + 94.9x_{DH} + 88.9x_{DI})$$

given

Attleboro:	$x_{AG} + x_{AH} + x_{AI} = 6000$
Boxborough:	$x_{BG} + x_{BH} + x_{BI} = 6000$
Chelmsford:	$x_{CG} + x_{CH} + x_{CI} = 6000$
Dartmouth:	$x_{DG} + x_{DH} + x_{DI} = 6000$
Gloucester:	$x_{AG} + x_{BG} + x_{CG} + x_{DG} \leq 9000$
Haverhill:	$x_{AH} + x_{BH} + x_{CH} + x_{DH} \leq 9000$
Ipswich:	$x_{AI} + x_{BI} + x_{CI} + x_{DI} \leq 9000$
Route maximum:	$x_{XY} \leq 5000y_{XY}$
Route minimum:	$2000y_{XY} \leq x_{XY}$

The final two constraints, "Route maximum" and "Route minimum," apply to all 12 possible values of X and Y, so that there are actually 24 constraints to impose.

Excel and *Solver* report that the optimal solution is to ship 4,000 gallons from Attleboro to Gloucester, 2,000 gallons from Attleboro to Ipswich, 4,000 gallons from Boxborough to Haverhill, 2,000 gallons from Boxborough to Ipswich, 4,000 gallons from Chelmsford to Haverhill, 2,000 gallons from Chelmsford to Ipswich, 3,000 gallons from Dartmouth to Gloucester, and 3,000 gallons from Dartmouth to Ipswich. The total cost is $13,818.

2. **Barton's Groceries** is contracting with different vendors to replace the roof at each of their four stores. Barton's has received six bids, and has decided to hire *exactly three* vendors to repair the four roofs. The bids are:

	Location			
Bidder	1	2	3	4
A	190	175	125	230
B	150	235	155	220
C	210	225	135	260
D	170	185	190	280
E	220	190	140	240
F	270	200	130	260

(*a*) Formulate a linear model to help Barton's choose the three roofers while minimizing total cost

(*b*) Solve your model using *Excel* and *Solver*.

Answer: Let

$$y_{ij} = \begin{cases} 1 & \text{if bidder } i \text{ is accepted for location } j \\ 0 & \text{if bidder } i \text{ is } \textit{not} \text{ accepted for location } j \end{cases}$$

$$z_i = \begin{cases} 1 & \text{if bidder } i \text{ is accepted for any bids} \\ 0 & \text{if bidder } i \text{ is } \textit{not} \text{ accepted for any bids} \end{cases}$$

where $i = A$, B, C, D, E, or F, and $j = 1$, 2, 3, or 4. We need to minimize

$$190y_{A1} + y_{A2}175 + 125y_{A3} + 230y_{A4} + 150y_{B1} + 235y_{B2} + 155y_{B3} + 220y_{B4}$$
$$+$$
$$210y_{C1} + 225y_{C2} + 135y_{C3} + 260y_{C4} + 170y_{D1} + 185y_{D2} + 190y_{D3} + 280y_{D4}$$
$$+$$
$$220y_{E1} + 190y_{E2} + 140y_{E3} + 240y_{E4} + 270y_{F1} + 200y_{F2} + 130y_{F3} + 260y_{F4}$$

given

Location 1: $y_{A1} + y_{B1} + y_{C1} + y_{D1} + y_{E1} + y_{F1} = 1$

Location 2: $y_{A2} + y_{B2} + y_{C2} + y_{D2} + y_{E2} + y_{F2} = 1$

Location 3: $y_{A3} + y_{B3} + y_{C3} + y_{D3} + y_{E3} + y_{F3} = 1$

Location 4: $y_{A4} + y_{B4} + y_{C4} + y_{D4} + y_{E4} + y_{F4} = 1$

Bidder A: $y_{A1} + y_{A2} + y_{A3} + y_{A4} \leq 4z_A$

$$z_A \leq y_{A1} + y_{A2} + y_{A3} + y_{A4}$$

Bidder B: $y_{B1} + y_{B2} + y_{B3} + y_{B4} \leq 4z_B$

$$z_B \le y_{B1} + y_{B2} + y_{B3} + y_{B4}$$

Bidder C: $$y_{C1} + y_{C2} + y_{C3} + y_{C4} \le 4z_C$$

$$z_C \le y_{C1} + y_{C2} + y_{C3} + y_{C4}$$

Bidder D: $$y_{D1} + y_{D2} + y_{D3} + y_{D4} \le 4z_D$$

$$z_D \le y_{D1} + y_{D2} + y_{D3} + y_{D4}$$

Bidder E: $$y_{E1} + y_{E2} + y_{E3} + y_{E4} \le 4z_E$$

$$z_E \le y_{E1} + y_{E2} + y_{E3} + y_{E4}$$

Bidder F: $$y_{F1} + y_{F2} + y_{F3} + y_{F4} \le 4z_F$$

$$z_F \le y_{F1} + y_{F2} + y_{F3} + y_{F4}$$

Choose 3 bidders: $$z_A + z_B + z_C + z_D + z_D + z_F = 3$$

Because this is a transportation problem, it actually is unnecessary to add the constraints that y_{ij} are binary variables, though it is not wrong to do so.

Excel and *Solver* report that the optimal solution is to hire bidder A for location 2, bidder B for locations 1 and 4, and bidder F for location 3, for a total cost of \$675.

3. **Zugzwang Manufacturing** has received an order to make 150 chess sets. Zugzwang has two different assembly lines, either or both of which can be used to fulfill the order. Each assembly line has a set-up cost associated to its use, each has a different cost per chess set manufactured, and each uses a different amount of labor to manufacture the sets. The details:

Assembly line	Set-up cost	Manufacturing cost/set	Labor required/set
A	\$25	\$0.13	15 minutes
B	\$30	\$0.12	30 minutes

Up to 40 hours of labor are available at the rate of \$9/hour. These hours can be distributed between the two different assembly lines freely. Each assembly line must manufacture an integer number of chess sets.

(a) Formulate a linear program to help Zugzwang manufacture the sets at minimal cost. Define all decision variables clearly, state the objective function, and list all constraints. All 150 sets must be manufactured.

(b) Use *Excel* and *Solver* to solve the problem.

Suppose that Zugzwang gets an order for an additional 25 sets, so that a total of 175 must be manufactured. Zugzwang can use up to 10 hours of overtime, at the rate of \$11/hour. Union rules require that if any overtime is used, a minimum of 5 hours must paid for.

(c) Formulate a linear program to solve the problem including the possibility of overtime. Define all new decision variables, state the modified objective function, and list all new and modified constraints.

(d) Use *Excel* and *Solver* to solve the modified problem.

(e) (*No credit*) What is the meaning of *Zugzwang*?

Answer: (a) Let

$$a = \text{number of sets manufactured on assembly line A}$$

$$b = \text{number of sets manufactured on assembly line B}$$

$$y_a = \begin{cases} 1 & \text{if assembly line A is used} \\ 0 & \text{if assembly line A is } \textit{not} \text{ used} \end{cases}$$

$$y_b = \begin{cases} 1 & \text{if assembly line B is used} \\ 0 & \text{if assembly line B is } \textit{not} \text{ used} \end{cases}$$

We need to minimize

$$25y_a + 30y_b + 0.13a + 0.12b + 9(.25a + .50b)$$

given that a and b are integers, and

Order size: $a + b = 150$

Turn binary variables on and off:

$$a \le 150y_a$$
$$y_a \le a$$
$$b \le 150y_b$$
$$y_b \le b$$

Labor hours: $0.25a + 0.50b \le 40$

(b) *Excel* and *Solver* report that the optimal solution is to manufacture 150 sets using assembly line A, for a total cost of $382.

(c) The changes make the problem considerably more complicated. Let

$$r = \text{number of hours of labor at normal rate}$$
$$t = \text{number of hours of labor at overtime rate}$$

$$y_t = \begin{cases} 1 & \text{if overtime is used} \\ 0 & \text{if overtime is } \textit{not} \text{ used} \end{cases}$$

The objective is now to minimize:

$$25y_a + 30y_b + 0.13a + 0.12b + 9r + 11t$$

The labor hour constraint must be rewritten, and we also must add constraints involving r and t:

Order size: $a + b = 175$

Labor hours: $0.25a + 0.50b \le r + t$

Maximum regular time: $r \le 40$

Maximum overtime: $t \le 10y_t$

Minimum overtime: $5y_t \le t$

Only use overtime after 40 hours: $40y_t \le r$

The solution is to manufacture all 175 chess sets on assembly line A. Even though only 3.75 hours of overtime are needed, 5 hours must be paid for, and the total cost is $462.75.