Mathematics 235
Robert Gross
Homework 8
Answers

1. **Burnside Marketing Research** conducted a study to design a new breakfast cereal. Three attributes were found to be significant: ratio of wheat to corn, type of sweetener (sugar, honey, or artificial), and the presence or absence of raisins. Seven children participated in taste tests and provided the following point values for their preferences:

	Wheat/Corn		**Sweetener**			**Raisins**	
Child	**Low**	**High**	**Sugar**	**Honey**	**Artificial**	**Present**	**Absent**
1	15	35	30	40	25	15	9
2	30	20	40	35	35	8	11
3	40	25	20	40	10	7	14
4	35	30	25	20	30	15	18
5	25	40	40	20	35	18	14
6	20	25	20	35	30	9	16
7	30	15	25	40	40	20	11

(*a*) Formulate a linear model that produces a cereal recipe with the highest average score among the seven children.

(*b*) Solve using *Excel* and *Solver*. What is the cereal recipe with the highest average score?

(*c*) The current leading brand has a **low** ratio of wheat to corn, is sweetened with **honey**, and does **not** have raisins. Formulate a linear model for a cereal that will score at least 1 point higher with as many children as possible. The objective function is the number of children who will prefer the new cereal to the current leading brand.

(*d*) Solve your model using *Excel* and *Solver*. What is the cereal recipe that will induce the maximum number of children to change brands?

(*e*) Change the spreadsheet to ask for a score that will be at least 5 points higher than the current leading brand. Solve this reformulated problem. What is the recipe for the new cereal that will have a score at least 5 points higher with as many children as possible?

Answer: Number the attributes (wheat-to-corn ratio, sweetener used, and raisins) from 1 to 7. Define the 7 binary variables

$$y_i = \begin{cases} 1 & \text{if attribute } i \text{ is chosen} \\ 0 & \text{if attribute } i \text{ is not chosen} \end{cases}$$

We need to maximize the average score, which is

$$\begin{aligned} \frac{1}{7}\Big(& 5y_1 + 35y_2 + 30y_3 + 40y_4 + 25y_5 + 15y_6 + 9y_7 + \\ & 30y_1 + 20y_2 + 40y_3 + 35y_4 + 35y_5 + 8y_6 + 11y_7 + \\ & 40y_1 + 25y_2 + 20y_3 + 40y_4 + 10y_5 + 7y_6 + 14y_7 + \\ & 35y_1 + 30y_2 + 25y_3 + 20y_4 + 30y_5 + 15y_6 + 18y_7 + \\ & 25y_1 + 40y_2 + 40y_3 + 20y_4 + 35y_5 + 18y_6 + 14y_7 + \\ & 20y_1 + 25y_2 + 20y_3 + 35y_4 + 30y_5 + 9y_6 + 16y_7 + \end{aligned}$$

$$30y_1 + 15y_2 + 25y_3 + 40y_4 + 40y_5 + 20y_6 + 11y_7\Big).$$

The constraints are

Wheat/corn ratio:	$y_1 + y_2 = 1$
Sweetener:	$y_3 + y_4 + y_5 = 1$
Raisins:	$y_6 + y_7 = 1$

Excel and *Solver* report that the highest average score comes from a recipe using a low wheat/corn ratio, honey, and no raisins, and the score is 74.

Not surprisingly, this is the recipe that the competition uses. Suppose that want to induce as many children as possible to change their preference. We need to add binary variables

$$z_i = \begin{cases} 1 & \text{if child } i \text{ switches} \\ 0 & \text{if child } i \text{ does not switch} \end{cases}$$

Now, we want to maximize $z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7$ given the above constraints and

Child 1:	$5y_1 + 35y_2 + 30y_3 + 40y_4 + 25y_5 + 15y_6 + 9y_7 \geq 64z_1 + 1$
Child 2:	$30y_1 + 20y_2 + 40y_3 + 35y_4 + 35y_5 + 8y_6 + 11y_7 \geq 76z_2 + 1$
Child 3:	$40y_1 + 25y_2 + 20y_3 + 40y_4 + 10y_5 + 7y_6 + 14y_7 \geq 94z_3 + 1$
Child 4:	$35y_1 + 30y_2 + 25y_3 + 20y_4 + 30y_5 + 15y_6 + 18y_7 \geq 73z_4 + 1$
Child 5:	$25y_1 + 40y_2 + 40y_3 + 20y_4 + 35y_5 + 18y_6 + 14y_7 \geq 59z_5 + 1$
Child 6:	$20y_1 + 25y_2 + 20y_3 + 35y_4 + 30y_5 + 9y_6 + 16y_7 \geq 71z_6 + 1$
Child 7:	$30y_1 + 15y_2 + 25y_3 + 40y_4 + 40y_5 + 20y_6 + 11y_7 \geq 81z_7 + 1$

The numbers on the right-hand side of the inequality (64, 76, etc.) are the scores that the children give to the current leading brand. *Excel* and *Solver* report that the solution is to formulate a cereal with high wheat/corn ratio, honey, and no raisins. Children 1, 5, and 6 will switch with this recipe.

Now we can change the "+1" on the right-hand side of the inequalities to "+5", and solve the problem again. *Excel* and *Solver* report that the optimal recipe now calls for low wheat/corn ratio, sugar, and no raisins. With this recipe, children 2, 4, and 5 will switch.

Note that this is also a solution to the previous part of the problem. In fact, there are multiple recipes which will convince 3 of the children to switch.

2. **Green's Hardware** expects to sell 75,000 bolts annually. The cost of ordering bolts is \$15. Inventory cost is \$0.01/bolt/year. Suppose that the unit cost is \$0.15/bolt. What order size minimizes Green's annual costs? Formulate and solve using calculus. You need not assume that the order size is an integer. You might want to check your answer using *Excel* and *Solver*, but you need not submit that part of your solution.

Answer: Suppose that the order size is x, with $1 \leq x \leq 75000$. The number of orders annually is $75000/x$, so the annual ordering cost is $15(75000)/x = 1125000/x$. The inventory size is assumed to be $x/2$, so the annual inventory cost is $(x/2)(0.01) = 0.005x$. The unit cost is actually irrelevant, as we cannot control it, but we will include it. The annual cost of materials is $(0.15)(75000) = 11250$. We need to minimize

$$C(x) = 11250 + 0.005x + \frac{1125000}{x}.$$

We check the endpoints, and find that $C(1) = 1{,}136{,}250.005$ and $C(75000) = 11{,}640$. We also compute $C'(x) = 0.005 - \frac{1125000}{x^2}$. Solving for $C'(x) = 0$, we get $x = \sqrt{\frac{1125000}{0.005}} = 15000$. We compute $C(15000) = 11{,}400$, and therefore the minimum cost occurs with an order size of 15,000, with 5 annual orders.

3. Suppose that $f(x) = 3x^4 + 4x^3 - 36x^2 + 5$.
   - (a) Find all critical values of $f(x)$, and use the second derivative test to determine whether a critical value is a local maximum or a local minimum.
   - (b) Suppose that we impose the restriction that $1 \le x \le 5$. Find the largest and smallest values that $f(x)$ can take.

Answer: We compute

$$f'(x) = 12x^3 + 12x^2 - 72x$$
$$f''(x) = 36x^2 + 24x - 72$$

We see that $f'(x) = 12x(x^2 + x - 6) = 12x(x+3)(x-2)$, and therefore the critical values are $x = 0$, $x = -3$, and $x = 2$. We compute $f''(-3) = 180$, $f''(0) = -72$, and $f''(2) = 120$. Therefore, -3 is a relative minimum, 0 is a relative maximum, and 2 is a relative minimum.

If the values of x are restricted to be between 1 and 5, then the extreme values occur either at $x = 1$, $x = 5$, or at any critical points in the interval, which in this case means $x = 2$. We compute $f(1) = -24$, $f(2) = -59$, and $f(5) = 1480$. Therefore, the largest value for $f(x)$ is 1480, and the smallest is -59.

4. **Tremont Concrete** orders 20,000 pounds of sand annually. The cost of placing an order is \$20. Inventory cost is 1%/lb/year of the unit cost of a pound of sand.
   - (a) Suppose that the cost of sand is \$1.30/lb. What order size minimizes Tremont's total cost? Solve using calculus. Orders need not be for a whole number of pounds.
   - (b) Suppose now that there is a quantity discount. The price is \$1.30/lb on orders of at most 5,000 pounds, but on orders of 5,001 pounds or more, the price is \$1.20/lb. To clarify, this means that the price of ordering 5,001 pounds is (\$1.20)(5,001)=\$6,001.20. Now find the order size that minimizes Tremont's total cost. Again, minimize using calculus.
   - (c) More realistically, suppose that the quantity discount works so that the first 5,000 pounds of any order cost \$1.30/lb, and any additional pounds of sand cost \$1.20. Now an order of 5,001 pounds costs (\$1.30)(5,000) + (\$1.20)(1) = \$6,501.20. What order size minimizes total cost?

Answer: Let x be the order size. The annual number of orders is $20000/x$. We assume as usual that the inventory size is $x/2$.

(a) The annual ordering cost is $(20000/x)(20) = 400000/x$. The annual material cost is $(20000)(1.3) = 26000$. The annual inventory cost is $(x/2)(1.30)(0.01) = 0.0065x$. We therefore need to minimize $C(x) = \frac{400000}{x} + 26000 + 0.0065x$, with $1 \le x \le 20000$. We compute $C'(x) = 0.0065 - 400000/x^2$, and solving for $C'(x) = 0$ yields $x = \sqrt{\frac{400000}{0.0065}} \approx 7844.6454$. We compute $C(7844.6454) \approx 26101.98$, $C(1) \approx 426000$, and $C(20000) = 26150$. Therefore, the minimum cost occurs with an order of 7844.6454, and the annual number of orders is approximately 2.5495.

(b) The computations in part (a) are still valid if $1 \le x \le 5000$. We can compute $C(5000) = 26112.5$. Now, if $5001 \le x \le 2000$, the annual ordering cost is still $400000/x$, but the material cost is now $(20000)(1.2) = 24000$, and the annual inventory cost is $(x/2)(1.20)(0.01) = 0.006x$. Therefore, $C(x) = \frac{400000}{x} + 24000 + 0.006x$. We compute $C'(x) = 0.006 - \frac{400000}{x^2}$. Solving $C'(x) = 0$ yields $x = \sqrt{\frac{400000}{0.006}} \approx 8164.9658$. We compute $C(5001) = 24109.99$, $C(8164.9658) \approx 24097.9796$, and $C(20000) = 24140$. Therefore, the minimum cost occurs with an order size of 8164.9658, with 2.4495 orders annually.

(c) For $1 \leq x \leq 5000$, the computations above are still valid, and the minimum cost occurs when $x = 5000$, with $C(5000) = 26112.5$.

Now, suppose that $5001 \leq x \leq 20000$. The material cost in each order is $1.3 \cdot 5000 + 1.2(x - 5000) = 1.2x + 500$, and the number of orders is $20000/x$, so the annual material cost is $(20000/x)(1.2x + 500) = \frac{10000000}{x} + 24000$. The annual ordering cost is still $400000/x$. The cost of the material in inventory is $0.6x + 250$ (because we assume that on average inventory is half of the order size), and therefore the cost of inventory is $(0.01)(0.6x + 250) = 0.006x + 2.5$. Our cost function is $C(x) = \frac{10000000}{x} + 24000 + \frac{400000}{x} + 0.006x + 2.5 = 0.006x + \frac{10400000}{x} + 24002.5$. We compute $C'(x) = 0.006 - \frac{10400000}{x^2}$, and solving for $x = 0$ yields $x = \sqrt{\frac{10400000}{0.006}} \approx 41633.32$, which is outside the range permitted. We compute $C(5001) \approx 26112.0901$ and $C(20000) \approx 24642.5$. Therefore the minimal cost occurs if we place one order of 20000 pounds annually.

5. Let $f(x, y) = 2x^2 + 3xy + y^2 - 11x - 8y + 14$. Find all critical pairs for $f(x, y)$, and use the second derivative test to classify each critical pair as a local minimum, a local maximum, or a saddle point.

Answer: We compute

$$\begin{aligned} f_x &= 4x + 3y - 11 \\ f_y &= 3x + 2y - 8 \\ f_{xx} &= 4 \\ f_{xy} &= 3 \\ f_{yy} &= 2 \end{aligned}$$

To find critical values, we simultaneously solve $4x + 3y - 11 = 0$ and $3x + 2y - 8 = 0$, and find that $x = 2$ and $y = 1$. We compute $D = f_{xx}f_{yy} - f_{xy}^2 = 4 \cdot 2 - 3^2 < 0$, and therefore $(2, 1)$ is a saddle point.