Mathematics 235
Robert Gross
Homework 9
Answers

1. A certain port has 3 loading docks, labelled A, B, and C. The distances between the docks, in meters, is:

	A	B	C
A	–	100	175
B	100	–	75
C	175	75	–

Three boats, the *Nina*, the *Pinta*, and the *Santa Maria*, need to come into port and dock simultaneously, one at each dock, and transfer goods. The quantity of cargo to be transferred, in kilograms, is

	To		
From	*Nina*	*Pinta*	*Santa Maria*
Nina	–	60	75
Pinta	–	–	80
Santa Maria	35	85	–

The objective is to minimize the product of the weight of the cargo that must be moved and the distance that the cargo is moved. For example, if the *Nina* is in dock A, the *Pinta* in dock B, and the *Santa Maria* in dock C, then the product will be $60 \cdot 100 + 75 \cdot 175 + 80 \cdot 75 + 175 \cdot 35 + 75 \cdot 85$.

Formulate this problem, which probably will involve binary variables. Be sure to state your objective function and constraints clearly. The objective function will probably not be linear. Then use *Excel* and *Solver* to find the optimal solution.

Answer: We start by defining 9 binary variables. Let

$$y_{NA} = \begin{cases} 1 & \text{if the } \textit{Nina} \text{ is in dock } A \\ 0 & \text{otherwise} \end{cases}$$

and similarly for y_{NB}, y_{NC}, y_{PA}, y_{PB}, y_{PC}, y_{SA}, y_{SB}, and y_{SC}. There are 6 constraints, to make sure that a boat is in only one dock and each dock contains only one boat:

Nina: $$y_{NA} + y_{NB} + y_{NC} = 1$$
Pinta: $$y_{PA} + y_{PB} + y_{PC} = 1$$
Santa Maria: $$y_{SA} + y_{SB} + y_{SC} = 1$$
Dock A: $$y_{NA} + y_{PA} + y_{SA} = 1$$
Dock B: $$y_{NB} + y_{PB} + y_{SB} = 1$$
Dock C: $$y_{NC} + y_{PC} + y_{SC} = 1$$

The objective function is trickier to formulate. Start by noticing that it does not matter whether cargo is moved from the *Nina* to the *Pinta* or *vice versa*, and similarly for any other two choices of boats. The relevant weights are only the total amount of cargo moved between the two ships:

Ships	Cargo transferred (kg)
Nina–Pinta	60
Nina–Santa Maria	110
Pinta–Santa Maria	165

Now, suppose that the *Nina* is in dock A and the *Pinta* is in dock B, or *vice versa.* The distance between them is 100m, so this piece of the objective function looks like

$$100(y_{NA}y_{PB} + y_{NB}y_{PA})60.$$

If one of those two boats is in dock A and the other is in dock C, then another piece of the objective function is

$$175(y_{NA}y_{PC} + y_{NC}y_{PA})60.$$

And there is a third piece of the objective function, corresponding to docks B and C:

$$75(y_{NB}y_{PC} + y_{NC}y_{PB})60.$$

Adding, the part of the objective function covering the transfer of cargo from the *Nina* to the *Pinta* is

$$60\,(100(y_{NA}y_{PB} + y_{NB}y_{PA}) + 175(y_{NA}y_{PC} + y_{NC}y_{PA}) + 75(y_{NB}y_{PC} + y_{NC}y_{PB}))$$

Repeat this three times, and the entire objective function is:

Nina–Pinta: $$60\,(100(y_{NA}y_{PB} + y_{NB}y_{PA}) + 175(y_{NA}y_{PC} + y_{NC}y_{PA}) + 75(y_{NB}y_{PC} + y_{NC}y_{PB}))$$

$$+$$

Nina–Santa Maria: $$110\,(100(y_{NA}y_{SB} + y_{NB}y_{SA}) + 175(y_{NA}y_{SC} + y_{NC}y_{SA}) + 75(y_{NB}y_{SC} + y_{NC}y_{SB}))$$

$$+$$

Pinta–Santa Maria: $$165\,(100(y_{PA}y_{SB} + y_{PB}y_{SA}) + 175(y_{PA}y_{SC} + y_{PC}y_{SA}) + 75(y_{PB}y_{SC} + y_{PC}y_{SB}))$$

Excel and *Solver* report that the optimal solution is to dock the *Nina* in dock A, the *Santa Maria* in dock B, and the *Pinta* in dock C. The total expenditure is 33,875 kg-m.

2. The Cobb–Douglas production function is a classic model from economics used to model output as a function of capital and labor. It has the form

$$f(L, C) = aL^bC^d$$

where L represents units of labor, C represents units of capital, and a, b, and d are constants.

For example, set $a = 5$, $b = 0.25$ and $d = 0.75$, and assume that the cost of a unit of labor is \$25 and the cost of a unit of capital is \$75. Assume that a total of \$75,000 is available, and will all be used for either labor or capital.

(a) Write out a model to maximize $f(L, C)$. Be sure to state your constraints clearly. The model will be non-linear.

(b) Use *Excel* and *Solver* to find the optimal solution.

(c) What is the approximate shadow price for the constraint that a total of \$75,000 is available?

Answer: (a) We need to maximize $f(L, C) = 5L^{0.25}C^{0.75}$ given that $25L + 75C = 75000$, with $L \geq 0$ and $C \geq 0$.

(b) *Excel* and *Solver* report that the optimal solution is to use 750 units of labor and 750 units of capital, and the output will be 3,750. (We are not given units for the output.)

(c) The approximate shadow price for the constraint is given in the sensitivity report. It is the Lagrange multiplier, which is approximately 0.05.

3. Suppose that we need to find both the maximum and the minimum of the function $f(x, y) = 2x + 3y$, given the constraint $x^2 + y^2 = 10$.

(a) Formulate this problem using Lagrange multipliers. Write out the 3 equations involving x, y, and λ. Then solve the equations.

(b) Solve the problem using *Excel* and *Solver*. You will need to run *Solver* twice, once to find the maximum and once to find the minimum. Do not constraint the variables x and y to be positive.

(c) Verify that the numerical solutions found by *Solver* are the same as the ones that you found in part (a).

Answer: (a) We write $g(x, y) = x^2 + y^2 - 10$, and write out the equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ g(x, y) &= 0 \end{aligned}$$

We get

$$\begin{aligned} 2 &= \lambda(2x) \\ 3 &= \lambda(2y) \\ x^2 + y^2 &= 10 \end{aligned}$$

Divide the first equation by the second, and we get $\frac{2}{3} = \frac{x}{y}$, or $2y = 3x$. Therefore, $y = \frac{3}{2}x$. Substitute into the third equation, and we have $x^2 + \frac{9}{4}x^2 = 10$, or $\frac{13}{4}x^2 = 10$, or $x^2 = \frac{40}{13}$. Therefore, $x = \pm\sqrt{\frac{40}{13}} \approx \pm 1.7541$. Then $y = \frac{3}{2}x \approx \pm 2.6312$. The maximum value of the function occurs when $x \approx 1.7541$ and $y \approx 2.6312$, and the value of the function is approximately 11.4018. The minimum value of the function occurs when $x \approx -1.7541$ and $y \approx -2.6312$, and the value of the function is approximately -11.4018.

4. **Western Sports** sells winter clothing. The store must place its order for down vests well before the season starts, because they are made during the summer. Western must decide whether to place a large, medium, or small order for vests, and the number sold will depend mostly on whether the winter is mild, normal, or frigid.

The following table summarizes the net profit (in thousands of dollars) that Western expects under each scenario:

	Amount of Snow		
Size of order	Frigid	Normal	Mild
Large	10	7	3
Medium	9	9	7
Small	4	4	4

(a) Using the optimistic decision rule, what decision should Western Sports make? Explain your answer.
(b) Using the conservative decision rule, what decision should Western Sports make? Explain your answer.
(c) Use the minimax regret rule to help Western Sports decide. Explain your answer.
(d) Suppose that the probability of a frigid winter is 0.2, that of a normal winter is 0.5, and that of a mild winter is 0.3. Use the expected value approach to formulate a decision. What is the Expected Value of Perfect Information?

Answer: (a) The single biggest value in the table is 10, so using the optimistic approach, Western Sports would place a large order and hope for frigid weather.

(b) The conservative rule calls for a medium order, because that ensures a profit of no worse than 7 in all cases. A large order might result in a profit of 3, and a small order only returns a profit of 4.

(c) Let's add regrets to the table:

	Amount of Snow			*Regrets*			
Size of order	Frigid	Normal	Mild	Frigid	Normal	Mild	Max
Large	10	7	3	0	2	4	4
Medium	9	9	7	1	0	0	1
Small	4	4	4	6	5	3	3

Therefore, the minimax regret strategy calls for a medium order.

(d) The expected value of a large order is $10 \cdot 0.2 + 7 \cdot 0.5 + 3 \cdot 0.3 = 6.4$, the expected value of a medium order is 8.4, and the expected value of a small order is 4. The expected value approach calls for a medium order.

With perfect information, we would place a Large order with frigid weather, and a Medium order otherwise, and the expected value would be $10 \cdot 0.2 + 9 \cdot 0.5 + 7 \cdot 0.3 = 8.6$. Therefore, the expected value of perfect information is $8.6 - 8.4 = 0.2$ thousand dollars.

5. **Grow-It Farms** has a choice of four crops to plant in a certain field this month. The farm manager estimates the following yield in bushels, and incomes for each of the four crops, depending on the weather:

	Expected Yield			
Weather	**Crop 1**	**Crop 2**	**Crop 3**	**Crop 4**
Dry (D)	20	15	30	40
Moderate (M)	35	20	25	40
Rainy (R)	40	30	25	40
Income/bushel	\$1.00	\$1.50	\$1.00	\$0.80

(a) Using the optimistic decision rule, what decision should Grow-It make? Explain your answer.
(b) Using the conservative decision rule, what decision should Grow-It make? Explain your answer.
(c) Use the minimax regret rule to help Grow-It decide. Explain your answer.

(d) Suppose that the probability of a dry weather (D) is 0.10, that of moderate weather (M) is 0.55, and that of a rainy weather (R) is 0.35. Use the expected value approach to formulate a decision. What is the Expected Value of Perfect Information?

(e) Suppose that Grow-It can wait a week, and use the weather for that week to help estimate the various probabilities for the upcoming growing season. In particular, Grow-It will classify the rainfall during the coming week as high (H) or low (L), and use the following probabilities:

$$P(H) = 0.5300 \qquad P(L) = 0.4700$$
$$P(D|H) = 0.0189 \qquad P(D|L) = 0.1915$$
$$P(M|H) = 0.5189 \qquad P(M|L) = 0.5851$$
$$P(R|H) = 0.4623 \qquad P(R|L) = 0.2234$$

If Grow-It waits a week, what strategy should be followed? What is the Expected Value of Sample Information? What is the efficiency of the survey?

Answer: Let's start by converting the payoff table to dollars, because we wish to maximize income, not bushels:

	Expected Income			
Weather	**Crop 1**	**Crop 2**	**Crop 3**	**Crop 4**
Dry (D)	20.00	22.50	30.00	32.00
Moderate (M)	35.00	30.00	25.00	32.00
Rainy (R)	40.00	45.00	25.00	32.00

I could be clever and eliminate Crop 3 from consideration at this point, because in all circumstances Crop 4 is more profitable. I'll leave it in just for completeness, but it can never be the right option.

(a) The optimistic approach calls for planting Crop 2, because \$45 is the single largest number in the table.

(b) The pessimistic approach calls for planting Crop 4, because that guarantees a profit of \$32. If we planted Crop 1, we might end up with a profit of only \$20; if we planted Crop 2, we might end up with a profit of only \$22.50; and if we planted Crop 3, we might end up with a profit of only \$25.

(c) Let's add regrets to the table:

	Expected Income			
Weather	**Crop 1**	**Crop 2**	**Crop 3**	**Crop 4**
Dry (D)	20.00	22.50	30.00	32.00
Moderate (M)	35.00	30.00	25.00	32.00
Rainy (R)	40.00	45.00	25.00	32.00
	12.00	9.50	2.00	0.00
Regrets	0.00	5.00	10.00	3.00
	5.00	0.00	15.00	13.00
Maximum	12.00	9.50	20.00	13.00

Therefore, the minimax regret approach calls for planting Crop 2.

(d) The expected value of Crop 1 is 35.25, the expected value of Crop 2 is 34.50, the expected value of Crop 3 is 25.50, and the expected value of Crop 4 is 32.00. The expected value approach calls for planting Crop 1.

With perfect information, in dry weather we would plant Crop 4 with an income of 32, in moderate weather we would plant Crop 1 with an income of 35, and in rainy weather we would plant Crop 2 with an income of 45. The expected value with perfect information is 38.20, and the expected value of perfect information is $38.2 - 35.25 = 2.95$.

(e) If the rain this week is high, then the expected values of the four crops are respectively 37.0315, 36.7958, 25.0970, and 32. If the rainfall is high, we would plant Crop 1, with an expected value of 37.0315.

If the rain this week is low, then the expected values of the four crops are respectively 33.2445, 31.9148, 25.9575, and 32. If the rainfall is low, we would plant Crop 1, with an expected value of 33.2445.

The expected value of this approach is $37.0315 \cdot 0.53 + 33.2445 \cdot 0.47 = 35.25$. The expected value cannot change, because the approach is the same regardless of whether the rain was high or low. The expected value of sample information is 0, and the efficiency is also 0.

Decision Tree for Problem 5